

The number of Latin rectangles

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Abstract

We show how to generate an expression for the number of k -line Latin rectangles for any k . The computational complexity of the resulting expression, as measured by the number of additions and multiplications required to evaluate it, is on the order of $n^{(2^{k-1})}$. These expressions generalize Ryser's formula for derangements.

1 Was sind und was sollen die lateinische Rechtecken?

Let S be a set with n elements. A k -by- n matrix (A_{ij}) whose entries are drawn from the set S is called a *Latin rectangle* if no row or column of A contains a duplicate entry. Since the length of a row of the matrix A equals the size of the set S , each row must be a permutation of the set S . We could thus have described a Latin rectangle as a k -by- n matrix whose rows are mutually discordant permutations of the set S .

Examples:

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

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$$\begin{pmatrix} a & c & b & e & d \\ b & a & d & c & e \\ e & b & a & d & c \end{pmatrix}$$

The first example is a *Latin square*. Latin squares were investigated by Euler and are actually pretty interesting, as they are related to questions about finite projective planes. (See Ryser [3].) Latin rectangles are perhaps not so interesting, but they have the advantage of being easier to deal with.

Why *Latin*? Because, following Euler, we have chosen our set S to consist of letters from the Latin alphabet. If we had used Greek letters instead we would have had Greek rectangles:

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \delta & \gamma \end{pmatrix}$$

If, like Euler, we were to superimpose a Greek square and a Latin square, and if there were no repeated entries in the resulting square, then we would have our hands on a really interesting object called a *Graeco-Latin square*:

$$\begin{pmatrix} \alpha a & \beta b & \gamma c \\ \gamma b & \alpha c & \beta a \\ \beta c & \gamma a & \alpha b \end{pmatrix}$$

Many cheerful facts about such squares can be found in Ryser's book.

This being said, we immediately abandon the quaint custom of using letters for entries, and take for our n -element set S the integers from 1 to n :

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 & 5 & 4 \\ 2 & 1 & 4 & 3 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$$

Finally, we distinguish among all Latin rectangles those whose first row is in order. We call such rectangles *reduced*.

Examples:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

Any Latin rectangle can be reduced by permuting its columns, so that e.g. the unreduced 3-by-5 rectangle above gets reduced to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

2 The problem

Our object will be to find an expression for the number of k -line Latin rectangles. When we have done this we will say that we have “enumerated k -line Latin rectangles.”

Let us try to be more specific about what we mean by this. When we talk about “ k -line Latin rectangles,” the implication is that we are thinking of k as fixed and n as variable. To indicate this we denote the number of k -by- n Latin rectangles by $L_k(n)$. When we talk about “the number of k -line Latin rectangles”, we really mean the function L_k . And when we say that we want to “find an expression for the number of k -line Latin rectangles,” what we are looking for is an expression involving the variable n whose value upon substitution for n coincides with $L_k(n)$.

Contrast this with the problem of enumerating (just plain) Latin rectangles. If this were our object we would denote the number of k -by- n Latin rectangles by $L(k, n)$ to indicate that we were thinking of both k and n as variable, and we would look around for a single expression involving both k and n whose value upon substitution for k and n would coincide with $L(k, n)$.

Obviously if we could enumerate Latin rectangles we could enumerate k -line Latin rectangles for any k . Surprisingly, the converse of this statement is false. Thus, while we will be able to generate expressions for L_k for any k , and while it will even be clear how to write a computer program to generate these expressions, we won’t even have come close to enumerating Latin rectangles. This has to do with the dependence of the expressions for L_k on k . If we tried

to get around this by incorporating the process of generating the expression for L_k into a single expression involving k and n , we would find that the resulting expression was “not quite the kind of expression we had in mind ...”

At this point it would behoove us to say exactly what kind of expression we do have in mind. If we refrain from doing so, it is doubtless because we’re not really too clear on this point. Obviously certain expressions are no good, e.g.

$$\sum_{R \in \{1, \dots, n\}^{\{1, \dots, k\} \times \{1, \dots, n\}}} \chi_R$$

where

$$\chi_R = \begin{cases} 1 & \text{if } R \text{ is Latin} \\ 0 & \text{if not} \end{cases}$$

This example suggests one criterion we will expect an expression to meet, namely, that it take fewer operations to evaluate the expression than it would take to “check all cases.” Other criteria also suggest themselves, but nothing definitive. In any case the formulas we will produce for L_k turn out to be of an obviously “acceptable” form, so there is no need to go further into this question here.

In generating these formulas, our approach will be to generalize a formula for L_2 given by Ryser. I have recently learned that a fellow named James Nechvatal has also come up with formulas for the number of k -line Latin rectangles (Nechvatal [2]). Nechvatal’s method was quite different from the method we will be using, and the formulas he obtained bear no resemblance to ours.

Actually the formulas we will derive are formulas for $R_k(n)$, the number of reduced Latin rectangles, not formulas for $L_k(n)$. This is sufficient because $L_k(n) = n!R_k(n)$.

3 Ryser’s formula for derangements

A reduced 2-by- n rectangle is called a *derangement*, as it represents a permutation without fixed points.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix}$$

We can determine the number $D(n) = R_2(n)$ of derangements by beginning with the set of all permutations of the set $\{1, 2, \dots, n\}$ and “including-excluding” on the set of fixed points. (For a description of the method of inclusion-exclusion see Ryser [3].) Here’s what we get:

$$\begin{aligned}
 D(n) &= \text{total number of permutations of } \{1, 2, \dots, n\} \\
 &\quad - \sum_{\{i\}} \text{number of permutations fixing } i \\
 &\quad + \sum_{\{i, j\}} \text{number of permutations fixing } i \text{ and } j \\
 &\quad - \dots \\
 &= n! - n(n-1)! + \binom{n}{2}(n-2)! - \dots \\
 &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^n}{n!} \right).
 \end{aligned}$$

We write the formula in this way to emphasize that the ratio $D(n)/n!$, which represents the probability that a randomly selected permutation of $\{1, 2, \dots, n\}$ turns out to have no fixed points, is approaching

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots = \frac{1}{e}.$$

This formula for derangements has much to recommend it. However, in our enumeration we are going to be generalizing not this, but a second formula for the number of derangements:

$$D(n) = \sum_{r=0}^n (-1)^r \binom{n}{r} (n-r)^r (n-r-1)^{n-r}.$$

This second formula, due to Ryser, is also obtained from an inclusion-exclusion argument, though this new argument differs substantially from the argument above. In the next few sections we will present Ryser’s argument, not precisely as he presents it, but rather with an eye to generalizing it to rectangles with a larger number of rows.

4 Another way of looking at Latin rectangles

We begin by changing our conception of a Latin rectangle. To this end, let (A_{ij}) be a k -by- n Latin rectangle, and let

$$S_{ijl} = \begin{cases} 1 & \text{if } A_{ij} = l \\ 0 & \text{if not} \end{cases}$$

Evidently

1. $\sum_l S_{ijl} = 1$;
2. $\sum_j S_{ijl} \leq 1$ (no repeats in a row);
3. $\sum_i S_{ijl} \leq 1$ (no repeats in a column).

Conversely, any 0-1 valued “tensor” with these three properties arises from a Latin rectangle in this way. This gives us a new way of looking at a Latin rectangle.

If we think of taking a k -by- n -by- n block of cubes and selecting a subset of them of which S_{ijl} is the characteristic function, then we can rephrase conditions 1–3 above as follows:

1. there is exactly one block on any shaft;
2. there is at most one block on any hall;
3. there is at most one block on any corridor.

The terms “hall”, “corridor”, and “shaft” used here are illustrated in Figure 1. They come from imagining our pile of blocks to be a hotel, as in Figure 2. In the future we will frequently use this picture as a source of descriptive terminology. Thus e.g. when we talk about rooms at the back we will mean those cubes whose i -coordinate is 1, and when we say that two rooms are not on the same floor we will mean that they have different l -coordinates.

5 The idea behind the enumeration

Besides conditions 1–3 above there are a number of other similar ways of making sure that a selection of rooms determines a Latin rectangle. For instance when $k = n$, so that we are talking about Latin squares, we can phrase the requirement in the following more symmetrical way:

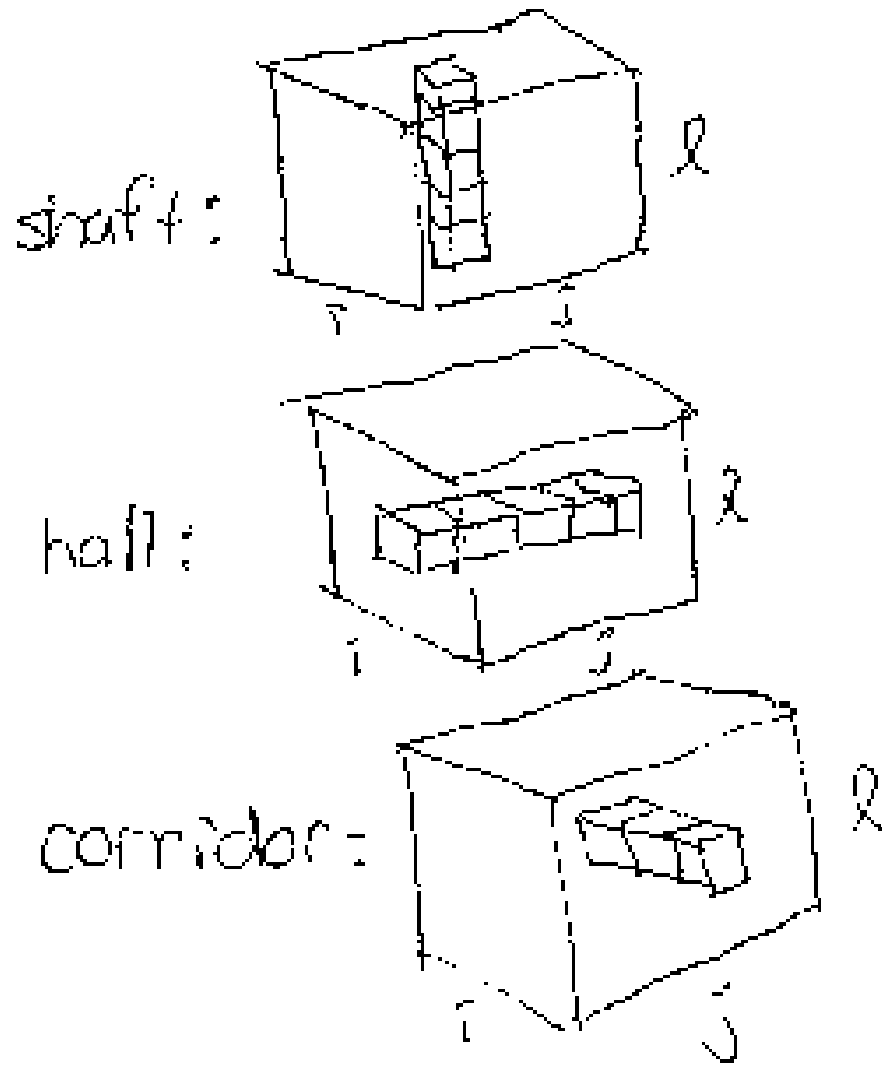


Figure 1: A shaft, a hall, a corridor.

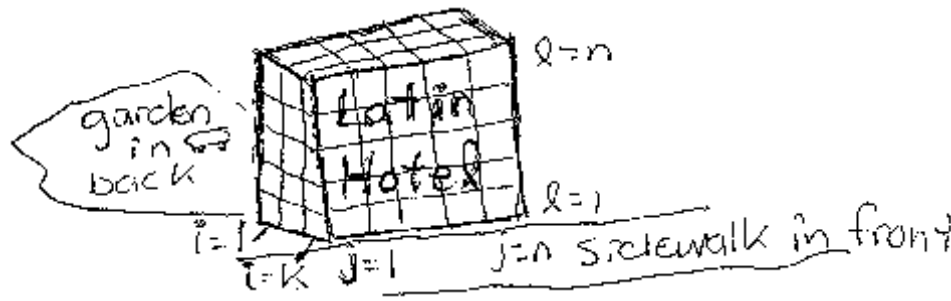


Figure 2: The Latin Hotel.

- there is exactly one room on any shaft, hall, or corridor.

In the case of a general rectangle, we will find it helpful to phrase the requirements as follows:

- there is *exactly* 1 room on any shaft;
- there is *at most* 1 room on any corridor;
- there is *at least* 1 room on any hall.

The idea will be to look at those configurations of rooms satisfying the first two conditions but possibly violating the third. For lack of a better term we will call such configurations *lonely-hall configurations* to indicate that there may be some halls that are not represented by our selection of rooms. The number of Latin rectangles is the number of lonely-hall configurations for which this term is a misnomer, i.e. for which the set of omitted halls is empty. We determine this number by inclusion-exclusion on the set of omitted halls.

Actually, the description just given does not quite fit what we are going to do, for in order to simplify our final formulas we will want to enumerate only reduced rectangles. Thus we will wind up looking at only those lonely-hall configurations having the standard “reduced” selection from the back halls, as shown in Figure 3. We will call such configurations *reduced lonely-hall configurations*, though it should be noted that it will not usually be possible to reduce an arbitrary lonely-hall configuration to a “reduced” one by interchanging columns.

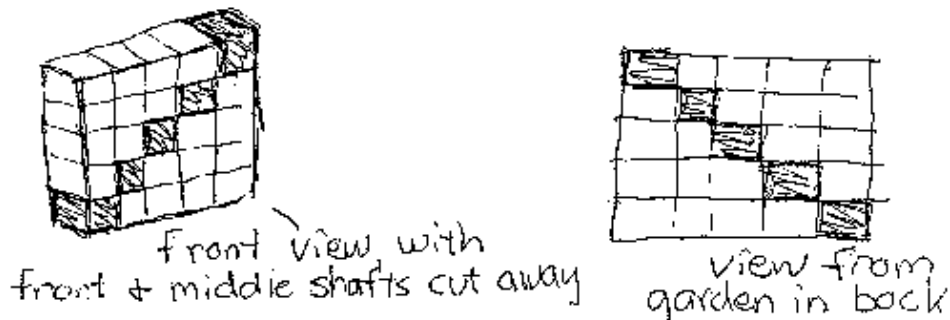


Figure 3: View of the back halls for a reduced lonely-hall configuration.

Again, we will want to use inclusion-exclusion on the omitted halls, but this time there will be no need to include the rear halls in the computation, as these will always be filled.

6 Derivation of Ryser's formula

In the case $k = 2$ we will only have to account for the n front halls in our inclusion-exclusion. To carry out the argument we ask ourselves:

- how many reduced lonely-hall configurations are there in all? *Answer:* $(n - 1)^n$
- of these, how many avoid a given front hall? *Answer:* $(n - 1)(n - 2)^{n-1}$
- how many avoid two given front halls? *Answer:* $(n - 2)^2(n - 3)^{n-3}$
- *etc.*

By inclusion-exclusion we get the number of selections leaving none of the front halls empty:

$$D(n) = R_2(n) = \sum_{r=0}^n (-1)^r \binom{n}{r} (n - r)^r (n - r - 1)^{n-r}.$$

This is Ryser's formula for derangements.

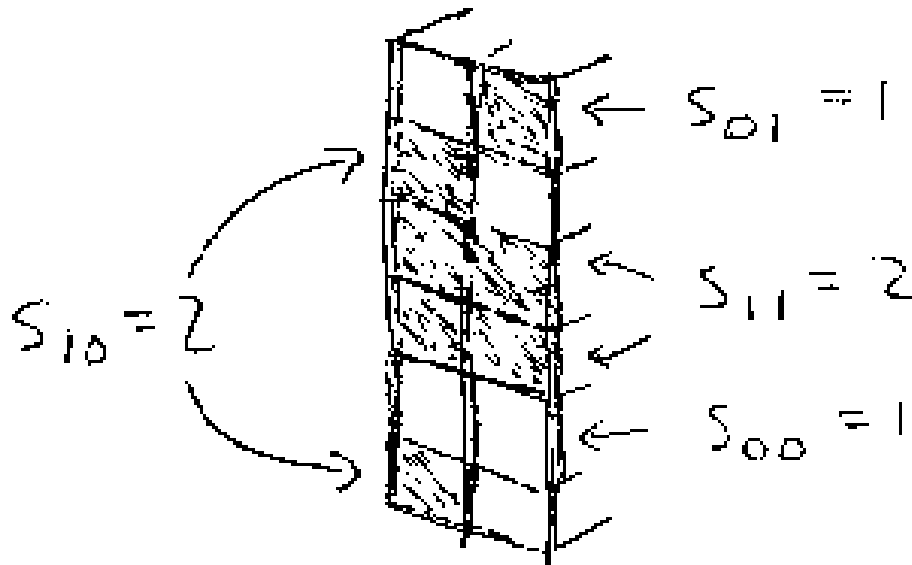


Figure 4: The parameters $s_{00}, s_{10}, s_{01}, s_{11}$. (Only the front and middle halls are shown.)

7 The number of 3-line Latin rectangles

In the case $k = 3$ we will have to include-exclude over halls at the front and middle of the hotel. Again what we need to know is the number $G(S)$ of lonely-hall configurations omitting a specified set S of front and middle halls. This number no longer depends only on the size of the set S . It turns out instead to depend on the four parameters $s_{00}, s_{10}, s_{01}, s_{11}$ defined as follows:

- s_{00} = the number of floors for which neither the middle nor the front hall belongs to S ;
- s_{10} = the number of floors for which the middle but not the front hall belongs to S ;
- s_{01} = ... the front but not the middle ...;
- s_{11} = ... both the front and the middle ...

This notation is illustrated in Figure 4.

Of course when n is fixed only 3 of these 4 quantities are independent, since

$$s_{00} + s_{10} + s_{01} + s_{11} = n.$$

Because $G(S)$ depends only on $(s_{00}, s_{10}, s_{01}, s_{11})$ we can write the inclusion-exclusion formula in the following form:

$$\begin{aligned} R_3(n) &= \sum_S (-1)^{|S|} G(S) \\ &= \sum_{s_{00}+s_{10}+s_{01}+s_{11}=n} (-1)^{s_{10}+s_{01}+2s_{11}} \binom{n}{s_{00}, s_{10}, s_{01}, s_{11}} G(s_{00}, s_{10}, s_{01}, s_{11}). \end{aligned}$$

All that remains to be done to finish the enumeration is to find an expression for the function G . We have been claiming that $G(S)$ depends only on $(s_{00}, s_{10}, s_{01}, s_{11})$ but in order not to get ahead of ourselves let us back off and think about how we would go about determining $G(S)$ if we didn't know this.

We are trying to determine the number of reduced lonely-hall configurations omitting all the halls in S . We can imagine that such a configuration is generated in the following way: We walk along the sidewalk in front of the hotel, and every time we see a new shaft of rooms towering above us we pick a room from that shaft and from the middle shaft directly behind it. As we pick these two rooms we make sure that our choices avoid the halls in S . and that together with the room in back already selected they represent 3 different floors. Evidently the n pairs of choices we make as we walk along may be made independently of one another. This means that $G(S)$ can be written as the product of n factors representing the number of choices we have in picking the n pairs of rooms.

In fact, if we weren't always having to worry about whether our choices interfere with the room already chosen in back we could write $G(S)$ as an n th power. The complication presented by the room in back is the price we have to pay for choosing to count reduced rectangles. We can try to repress this complication by pretending, as we choose each pair of rooms, that the set of halls we are trying to avoid is not S but

$$T = S \cup \{\text{halls on the same floor as the room already chosen in back}\}.$$

Then our problem reduces to determining the number $g(T)$ of ways of picking a front hall and a middle hall, not on the same floor, neither belonging to T . But this is easy:

$$g(T) = (t_{00} + t_{10})(t_{00} + t_{01}) - t_{00}$$

(choose the front room; choose the middle room; chuck the mess-ups).

Of course to go back from here and write down an expression for $G(S)$ we have to face up to the fact that the set T keeps changing as we proceed along the sidewalk. Luckily for us, while we may see as many as n different sets T in the course of our walk, we will see at most four different parameter sets $(t_{00}, t_{10}, t_{01}, t_{11})$. Since $g(T)$ depends only on these parameters, this enables us to write the following expression for $G(S)$:

$$G(S) = g(s_{00} - 1, s_{10}, s_{01}, s_{11} + 1)^{s_{00}} g(s_{00}, s_{10} - 1, s_{01}, s_{11} + 1)^{s_{10}} \\ \cdot g(s_{00}, s_{10}, s_{01} - 1, s_{11} + 1)^{s_{01}} g(s_{00}, s_{10}, s_{01}, s_{11})^{s_{11}}.$$

As promised, G depends only on $(s_{00}, s_{10}, s_{01}, s_{11})$.

Plugging our expression for G into the inclusion-exclusion formula above, we arrive at last at an expression for the number of 3-line Latin rectangles:

$$R_3(n) = \sum_{s_{00}+s_{10}+s_{01}+s_{11}=n} (-1)^{s_{10}+s_{01}+2s_{11}} \binom{n}{s_{00}, s_{10}, s_{01}, s_{11}} \\ \cdot g(s_{00} - 1, s_{10}, s_{01}, s_{11} + 1)^{s_{00}} g(s_{00}, s_{10} - 1, s_{01}, s_{11} + 1)^{s_{10}} \\ \cdot g(s_{00}, s_{10}, s_{01} - 1, s_{11} + 1)^{s_{01}} g(s_{00}, s_{10}, s_{01}, s_{11})^{s_{11}}.$$

where

$$g(t_{00}, t_{10}, t_{01}, t_{11}) = (t_{00} + t_{10})(t_{00} + t_{01}) - t_{00}.$$

This expression, for which we have struggled so valiantly, could hardly be called beautiful. Far prettier expressions for the number of 3-line rectangles are known. (Cf. Ryser [3], Bogart [1].) Its virtues are that it extends Ryser's formula for derangements, and that it does so in such a way as to make clear how to extend the enumeration to taller rectangles.

Before we take on higher values of k , let us say a few words about the computational complexity of the expression just obtained. We have expressed $R_3(n)$ as a triple sum. (It appears to be a 4-fold sum, but only 3 of the indices are independent.) Expanded out this sum has on the order of n^3 terms. A single term can be evaluated by performing a constant number of additions and something on the order of n multiplications. Thus the whole expression can be evaluated by performing something on the order of n^4 additions and multiplications.

8 Bigger values of k

At this point it should be clear how to write down an expression for $R_k(n)$ for any value of k . Here, for example, is the expression we would obtain for

the number of 4-line rectangles:

$$R_4(n) = \sum_{s_{000}+s_{100}+s_{010}+s_{001}+\dots+s_{111}=n} (-1)^{s_{100}+s_{010}+s_{001}+2s_{110}+\dots} \binom{n}{s_{000}, \dots, s_{111}} \cdot g_4(s_{000}-1, \dots, s_{111}+1)^{s_{000}} \cdot \dots \cdot g_4(s_{000}, \dots, s_{111})^{s_{111}},$$

where

$$g_4(t_{000}, \dots, t_{111}) = f_1 f_2 f_3 - f_{1,2} f_3 - f_{2,3} f_1 - f_{1,3} f_2 + 2f_{1,2,3},$$

where

$$\begin{aligned} f_1 &= t_{000} + t_{010} + t_{001} + t_{011}, \\ f_{1,2} &= t_{000} + t_{001}, \\ f_{1,2,3} &= t_{000}, \end{aligned}$$

and symmetrically for $f_2, f_3, f_{2,3}, f_{1,3}$.

Note that in writing down the expression for $g_4(t_{000}, \dots, t_{111})$ we have done a Möbius inversion in the lattice of partitions of a 3-set. Looking back at our expression for $g_3(t_{00}, \dots, t_{11})$, which we referred to simply as $g(t_{00}, \dots, t_{11})$, we see that that formula was obtained by a surreptitious Möbius inversion in the (3-element) lattice of partitions of a 2-set. Naturally to write down the formula for $R_k(n)$ we will need to know the formula for Möbius inversion in the lattice of partitions of a $(k-1)$ -set. Otherwise the procedure is completely straight-forward.

The expression we come up with will be a $(2^{k-1}-1)$ -fold sum containing on the order of $n^{2^{k-1}-1}$ terms. [Editor's note: The scanning software substituted "suck" for "sum" in the sentence above.] To evaluate each term will again take on the order of n multiplications and a constant number of additions. Thus it will be possible to evaluate the expression by making on the order of $n^{(2^{k-1})}$ additions and multiplications.

9 So what?

By now two things are clear:

- We could, if we wanted to, write a computer program that would ask for a value of k and respond by printing out an expression for $R_k(n)$.

- We are not likely to want to do this.

The reason is that the expression for $R_3(n)$ is bad enough, the expression for $R_4(n)$ is even worse, and the expressions get uglier and uglier at an exponential rate as k increases.

But while we are not likely to put these formulas under our pillow when we go to bed, we have at least shown that expressions for the number of k -line Latin rectangles can be found. And in the process we have gotten an idea of the computational complexity of the function $L_k(n)$.

10 Formulas for non-reduced rectangles

In our enumeration we chose to include-exclude over reduced lonely-hall configurations in order to reduce the complexity of the resulting formulas. At this point it may be worth while to go back and use our original idea of including-excluding over all lonely-hall configurations, just to see what happens. Here's what we get:

$$L_1(n) = n! = \sum_{r=0}^n (-1)^r \binom{n}{r} (n-r)^n,$$

$$L_2(n) = n!D(n) = \sum_{s_{00}+s_{10}+s_{01}+s_{11}=n} (-1)^{s_{10}+s_{01}+2s_{11}} \binom{n}{s_{00}, s_{10}, s_{01}, s_{11}} \cdot [(s_{00} + s_{10})(s_{00} + s_{01}) - s_{11}]^n,$$

$$L_3(n) = \sum_{s_{000}+\dots+s_{111}=n} (-1)^{s_{100}+2s_{110}+\dots} \binom{n}{s_{000}, \dots, s_{111}} \cdot [f_1 f_2 f_3 - f_{1,2} f_3 - f_{2,3} f_1 - f_{1,3} f_2 + 2f_{1,2,3}]^n,$$

where

$$\begin{aligned} f_1 &= s_{000} + s_{010} + s_{001} + s_{011}, \\ f_{1,2} &= s_{000} + s_{001}, \\ f_{1,2,3} &= s_{000}, \end{aligned}$$

and symmetrically for $f_2, f_3, f_{2,3}, f_{1,3}$.

These formulas, while perhaps in some way less “complicated” than the formulas for R_k , are much more complex. Whereas the expression for R_k could be evaluated by making on the order of $n^{(2^{k-1})}$ additions and multiplications, the formula for L_k is going to require more like $n^{(2^k)}$ additions and multiplications.

Not that we’re likely to be using either of these sets of formulas to make actual computations. When you come right down to it, no one really wants to know how many k -line Latin rectangles there are anyway.

References

- [1] K. P. Bogart. Counting 3-line Latin rectangles(?). ?, ?:?, ?
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- [3] H. J. Ryser. *Combinatorial Mathematics*. Mathematical Association of America, Washington, D. C., 1963.