

Laplace-isospectral hyperbolic 2-orbifolds are representation-equivalent

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Abstract

Let M be a compact hyperbolic 2-orbifold (not necessarily connected). We show that the spectrum of the Laplacian on functions on M determines the following: the volume; the total length of the mirror boundary; the number (properly counted) of cone-points of each order; and the number (properly counted) of primitive closed geodesics of each length and orientability class. This means that Laplace-isospectral hyperbolic 2-orbifolds are representation-equivalent, and hence strongly isospectral.

1 Statement

Let M be a compact hyperbolic 2-orbifold (not necessarily connected). Denote the eigenvalues of the Laplacian acting on functions on M by

$$0 = \lambda_0 \leq \lambda_1 \leq \dots$$

(Note that we don't necessarily have $0 < \lambda_1$ because we aren't assuming that M is connected.) We call the sequence $(\lambda_0, \lambda_1, \dots)$ the *Laplace spectrum* of

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M . If two spaces have the same Laplace spectrum we call them *Laplace-isospectral*. (Note that we don't simply call them 'isospectral', because this term is used in different ways by different authors.)

Our goal here will be to prove:

Theorem 1. *Let M be a compact hyperbolic 2-orbifold (not necessarily connected). The Laplace spectrum of M determines, and is determined by, the following data:*

1. *the volume;*
2. *the total length of the mirror boundary;*
3. *the number of cone-points of each order, counting a mirror corner as half a cone-point of the corresponding order;*
4. *the number of closed geodesics of each length and orientability class, counting a geodesic running along the boundary as half orientation-preserving and half orientation-reversing, and counting the k -fold iterate of a primitive geodesic as worth $\frac{1}{k}$ of a primitive geodesic of the same length and orientability.*

Of course the Laplace spectrum determines other data as well, for example the number of connected components. The data we list here determine those other data, since they determine the spectrum.

Theorem 1 can be recast less picturesquely as follows. Associated to a 2-orbifold M is a linear representation $\rho_{\bar{M}}$ of $\text{Isom}(H^2)$ on functions on the frame bundle \bar{M} of M . Associated to this representation is its character $\chi_{\bar{M}}$, a function on the set of conjugacy classes of $\text{Isom}(H^2)$. The geometrical data listed in Theorem 1 are just a way of describing geometrically the information conveyed by the character $\chi_{\bar{M}}$. The character determines the Laplace spectrum of M , and indeed, the spectrum of any other natural operator. Theorem 1 tells us that for hyperbolic 2-orbifolds, we can go back the other way, and recover the character from the Laplace spectrum. From the character, we get by general principles the linear equivalence class of the representation $\rho_{\bar{M}}$, and the spectrum of any natural operator on any natural bundle over M . Thus Laplace-isospectral hyperbolic 2-orbifolds are representation-equivalent and strongly isospectral. We will discuss these matters further in section 7 below; for now we concentrate on Theorem 1.

2 Examples

In this section, we give examples of the trade-offs between boundary and interior features that are allowed for in the statement of Theorem 1. Boundary corners on one orbifold may appear as conepoints on the other, while boundary geodesics may migrate to the interior. These examples should clarify just what it is we will be trying to prove, and why we can't expect to prove something stronger.

The examples here will be obtained by glueing bunches of congruent hyperbolic triangles. The glueing patterns arise from transplantable pairs, as described by Buser et al. [2].

Examples of trading boundary and interior geodesics are very common. (Indeed, practically any pair of isospectral hyperbolic 2-orbifolds with relecting boundary exhibit this phenomenon.) A variation on the famous example of Gordon, Webb, and Wolpert [9] yields a pair of planar hyperbolic 2-orbifolds of types $*224236$ and $*224623$, shown in Figure 1. This is likely the simplest pair of isospectral hyperbolic polygons, both in having the smallest number of vertices and having the smallest area. Each member of the pair is glued together from 7 copies of a prototype tile, according to the glueing pattern labelled 7_3 in [2]. The prototype tile is a so-called *346 triangle*, meaning a hyperbolic triangle with angles $\pi/3$, $\pi/4$, $\pi/6$.

There is no issue here with geodesics passing through the interior of any of the triangles that make up these two orbifolds: These can be matched so as to preserve length, orientability, and index of imprimitivity. But when it comes to geodesics that run along the edges of the triangles, whether along the mirror boundary or in the interior of the orbifold, it is necessary to balance boundary geodesics on one orbifold against interior geodesics on the other, as provided for in Theorem 1. To see this, look at Figure 1, and count geodesics on the two sides. Beware that boundary geodesics turn back at corners of even order, but continue along around the boundary at corners of odd order. Beware also of the way interior geodesics bounce when they hit the boundary. The resulting counts are indicated in Table 1. The names 'recto' and 'verso' are short for 'orientation-preserving' and 'orientation-reversing', in analogy with the names for the front and back of a printed page. The table only shows lengths for which there is at least one imprimitive geodesic. Trade-offs between boundary and interior geodesics continue at multiples of these lengths.

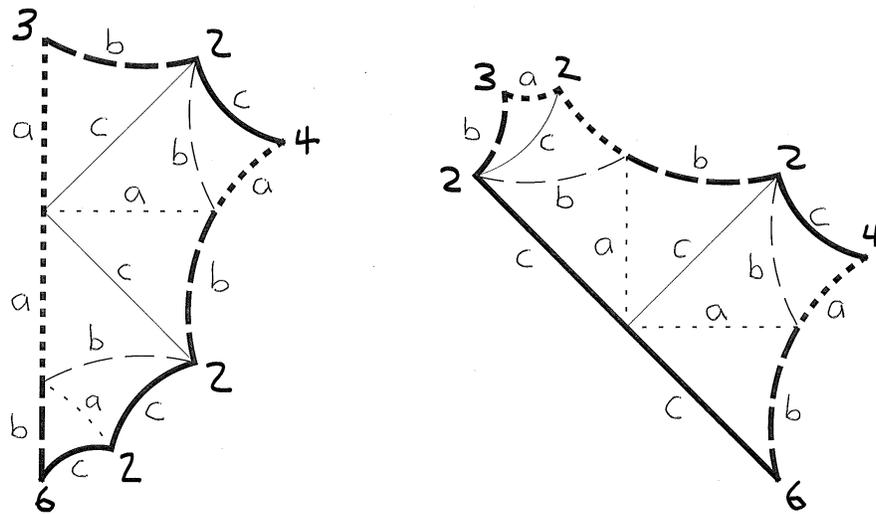


Figure 1: The pair $*224236$, $*224623$.

Geodesics for *224236:

length	boundary		interior		total	
	recto	verso	recto	verso	recto	verso
$2c$	$\frac{3}{2}$	$\frac{3}{2}$			$\frac{3}{2}$	$\frac{3}{2}$
$2a + 2b$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{3}{2}$	$\frac{3}{2}$
$4c$	$\frac{3}{2} \cdot \frac{1}{2}$	$\frac{3}{2} \cdot \frac{1}{2}$	1		$\frac{7}{4}$	$\frac{3}{4}$
$4a + 4b$	$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$	$2 \cdot \frac{1}{2}$		$\frac{7}{4}$	$\frac{3}{4}$

Geodesics for *224623:

length	boundary		interior		total	
	recto	verso	recto	verso	recto	verso
$2c$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{3}{2}$	$\frac{3}{2}$
$2a + 2b$	$\frac{3}{2}$	$\frac{3}{2}$			$\frac{3}{2}$	$\frac{3}{2}$
$4c$	$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$	$2 \cdot \frac{1}{2}$		$\frac{7}{4}$	$\frac{3}{4}$
$4a + 4b$	$\frac{3}{2} \cdot \frac{1}{2}$	$\frac{3}{2} \cdot \frac{1}{2}$	1		$\frac{7}{4}$	$\frac{3}{4}$

Table 1: Counting geodesics.

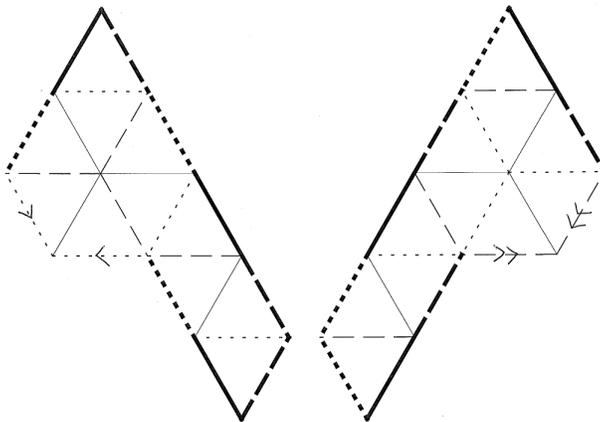


Figure 2: A transplanted pair of size 11.

Examples of trading corners for conepoints are not as common as examples of trading boundary and interior geodesics, but there are still plenty. Figures 2 and 3 show how to construct a pair $6 * 2232233$, $23 * 22366$. The transplanted pair used here is one of many found by John Conway using the theory of quilts. These will be described elsewhere, but there is no problem in verifying transplantedability here using the methods of [2]. Note how the order-6 conepoint moves to the boundary going one way, while the order-2 and order-3 conepoints move to the boundary going the other way.

Note. The groups associated to the prototype triangles in these two examples (346 and 366) both belong to the finite list of *arithmetic triangle groups*. (Cf. Maclachlan and Reid [13].) The finite-index subgroup of a triangle group that belongs to an orbifold obtained when the prototype triangle is arithmetic is also arithmetic. So our examples here are arithmetic.

3 Background

Huber [10] proved the result of Theorem 1 in the case of orientable hyperbolic 2-manifolds. Huber used what would nowadays be seen as a version of the Selberg trace formula, which allows us to read off the lengths of geodesics from the spectrum in a straight-forward way. Doyle and Rossetti [5] extended the result to non-orientable hyperbolic 2-manifolds. In the non-orientable case we can't simply read off the data about geodesics using the trace formula,

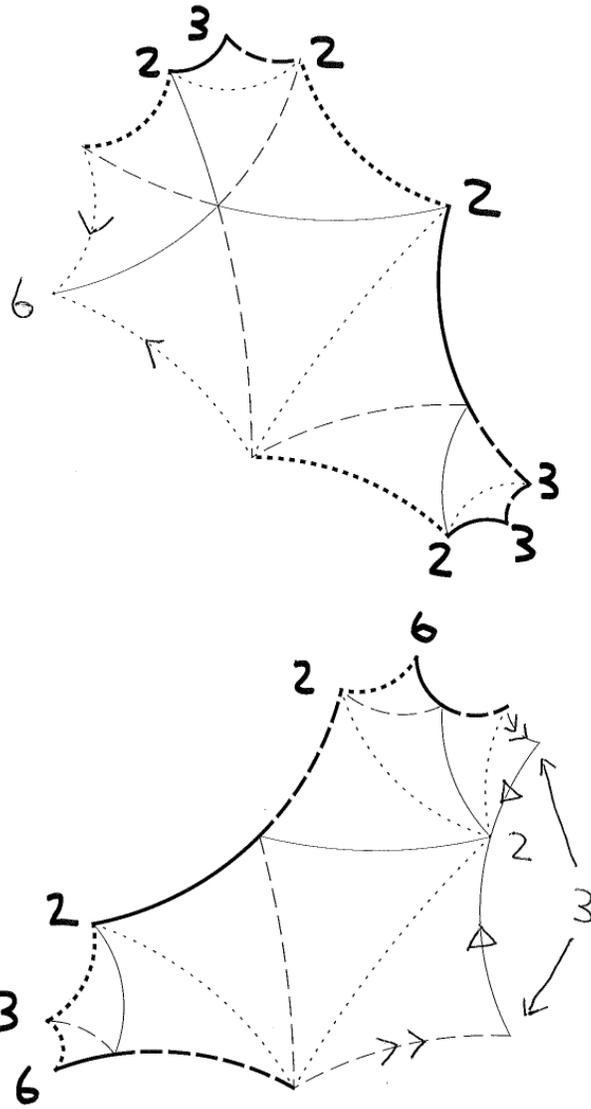


Figure 3: Replace the prototype equilateral triangle in Figure 2 by a 366 triangle so that six 3-vertices come together in the left-hand diagram. This yields a Laplace-isospectral pair of hyperbolic 2-orbifolds of types $6 * 2232233$ and $23 * 22366$.

because of interference between the spectral contributions of orientation-preserving and orientation-reversing geodesics of the same length. However, it turns out that any possible scenario for matching spectral contributions would require too many geodesics.

What we will show here is that, as we indicated in [5], it is a short step from non-orientable surfaces to general orbifolds. The reason is that the Selberg formula permits us to read off the data about orbifold features, just as Huber read off the data about geodesics in the case of orientable manifolds. Then we have only to check that there is still no scenario for matching the spectral contributions of the geodesics.

There are other ways to approach showing that the Laplace spectrum determines the data about orbifold features. By looking at the wave trace, Dryden and Strohmaier [8] proved the result of Theorem 1 for orientable hyperbolic 2-orbifolds. While they did not consider non-orientable surfaces or orbifolds, it seems likely that wave techniques could be used to show that the Laplace spectrum determines all the orbifold data. The argument would be essentially equivalent to the argument we give here, only more complicated.

Another possible approach is via the heat equation. By looking at short-time asymptotics of the heat trace, Dryden-Gordon-Greenwald-Webb [7] got information about the singular set of general orbifolds (for example, the volume of the reflecting boundary). Their results yield information about orbifolds of variable curvature, and in any dimension. Restricted to hyperbolic 2-orbifolds, the results they state don't yield complete information about the singular set. All this information is there in the short-time asymptotics of the heat trace, however, and presumably it could be extracted using their approach, by looking at higher and higher terms in the asymptotic expansion.

The beauty of the wave and heat approaches is that they can work in great generality. The Selberg method depends on having spaces whose underlying geometry is homogeneous, such as manifolds and orbifolds of constant curvature. For studying such spaces, it is a good bet that the Selberg method will beat the wave and heat approaches. Of course, the Selberg method can be used to treat the heat and wave kernels; the bet is that you will do better to consider a simpler kernel, like the counting kernel that we use here.

Finally, just as it is true that hyperbolic manifolds whose geodesics match for a sufficiently large number of lengths and twists (in particular, all but a finite number) must necessarily match for all lengths and twists (cf. Kelmer [11]), it is likely true on general principles that orbifolds with matching

geodesics must necessarily have matching orbifold features. That would make Theorem 1 follow almost immediately from the methods used to prove the corresponding result [5] about 2-manifolds, without explicitly computing the contributions of orbifold features to the spectrum, as we do here.

4 What we need from Selberg

Here we assemble what we will need from Selberg for the proof of Theorem 1. All the ideas come from Selberg [14].

Let $G = \text{Isom}(H^2)$ be the group of isometries of H^2 . Note that G has two components, corresponding to orientation-preserving and orientation-reversing isometries. A hyperbolic 2-orbifold can be written as a union of quotients of H^2 by discrete cocompact subgroups $\Gamma_j \subset G$, one for each connected component of M :

$$M = \cup_j \Gamma_j \backslash H^2.$$

We'll denote by $F(\Gamma_j)$ a fundamental domain for Γ_j .

In what follows, we could assume that M is connected, and write

$$M = \Gamma \backslash H^2,$$

because the extension to the disconnected case presents no difficulties. We prefer not to do this, in part to combat the common prejudice against disconnected spaces.

Define the *counting kernel* on H^2 by

$$c(x, y; s) = \begin{cases} 1, & d(x, y) \leq s \\ 0, & d(x, y) > s \end{cases}.$$

It tells when the hyperbolic distance $d(x, y)$ is at most s . Define the *counting trace*

$$\begin{aligned} C(s) &= \sum_j \int_{F(\Gamma_j)} \sum_{\gamma \in \Gamma_j} c(x, \gamma x; s) dx \\ &= \sum_j \int_{F(\Gamma_j)} \#\{\gamma \in \Gamma_j : d(x, \gamma x) \leq s\} dx. \end{aligned}$$

The counting trace tells (after dividing by the volume of M) the average number of broken geodesic loops on M of length at most s ,

The reason for the name ‘counting trace’ is that formally, $C(s)$ is the trace of the linear operator L_s whose kernel is the counting kernel $c(x, y; s)$ pushed down to M . L_s associates to a function $f : M \rightarrow \mathbb{R}$ the function whose value at $x \in M$ is the integral over a ball of radius s of the lift of f to the universal cover H^2 . L_s is not a actually trace class operator, because of the discontinuity of the counting kernel, but our expression for $C(s)$ is still well-defined.

We will need the following two propositions, which correspond to the two sides of the Selberg trace formula.

Proposition 1. *The Laplace spectrum determines the counting trace.* □

Let $\text{Cl}(\gamma, \Gamma)$ denote the conjugacy class of γ in Γ , and let $Z(\gamma, \Gamma)$ denote the centralizer. We have the usual one-to-one correspondence between the set of cosets $Z(\gamma, \Gamma) \backslash \Gamma$ and the conjugacy class $\text{Cl}(\gamma, \Gamma)$, where to the coset $Z(\gamma, \Gamma)\delta$ we associate the conjugate $\delta^{-1}\gamma\delta$.

Proposition 2. *The counting trace can be expressed as a sum of contributions from conjugacy classes of Γ :*

$$C(s) = \sum_j \sum_{\text{Cl}(\gamma, \Gamma_j)} \text{Vol}(\{x \in F(Z(\gamma, \Gamma_j)) : d(x, \gamma x) \leq s\}). \quad \square$$

The virtue of this proposition is that we can evaluate the summands using simple hyperbolic trigonometry.

5 Outline

The counting trace $C(s)$ is built up of contributions from the conjugacy classes of $\text{Isom}(H^2)$. The contribution from the identity is independent of s , and equal to the volume of M . The remaining conjugacy classes are of four kinds: reflections, rotations, translations, and glide reflections. Their contributions all vanish for $s = 0$, which means that $C(0)$ is the volume of M . Translations and glide reflections make no contribution for s smaller than the length of the shortest closed geodesic, so for small s the only contributions are from the identity, reflections, and rotations.

Begin by reading off the volume $C(0)$, and subtracting this constant from $C(s)$. In what is left, reflections make a contribution of first order in s , proportional to the total length of the mirror boundary. Rotations contribute only to second order, so we can read off the length of the mirror boundary, subtract out its contribution, and what is left is (for small s) entirely the result of rotations.

A simple computation now shows that the contributions of rotations through different angles are linearly independent. This allows us to read off the orders of the cone points, and subtract out their contributions. Once we've zapped the contributions of the identity, reflections, and rotations, what is left of the counting trace vanishes identically for small s , and overall is entirely the result of translations and glide reflections, corresponding in the quotient to closed geodesics.

Here we run into the main difficulty in the proof, which is that the contributions of orientation-preserving and orientation-reversing geodesics of a given length are not linearly independent. Indeed, they are proportional, with a constant of proportionality depending on the length of the geodesic. So we can't simply read off these lengths. Fortunately, we have already established that the spectrum determines the lengths and twists of geodesics for hyperbolic 2-manifolds, (See Doyle and Rossetti [5].) and the argument carries over directly to the orbifold case, providing we take care to count geodesics along the boundary as half orientation-preserving and half orientation-reversing.

6 Details

Denote by R the combined length of all reflecting boundaries.

Lemma 1. *The combined contribution of the reflecting boundaries to the counting trace $C(s)$ is*

$$R \sinh \frac{s}{2}.$$

Proof. Let $\rho \in \Gamma_j$ be a reflection. There are two possibilities for $Z(\rho, \Gamma_j) \backslash H^2$: a funnel or a planar domain. (See Figure 4.) In either case, the contribution to $C(s)$ is $R_\rho \sinh \frac{s}{2}$, where R_ρ is the portion of R attributable to $\text{Cl}(\rho, \Gamma_j)$, because this measures points in the quotient that are within $\frac{s}{2}$ of the axis of ρ , and hence within s of their images under the reflection ρ . (See Figure 5.) Summing over the conjugacy classes of reflections in all the

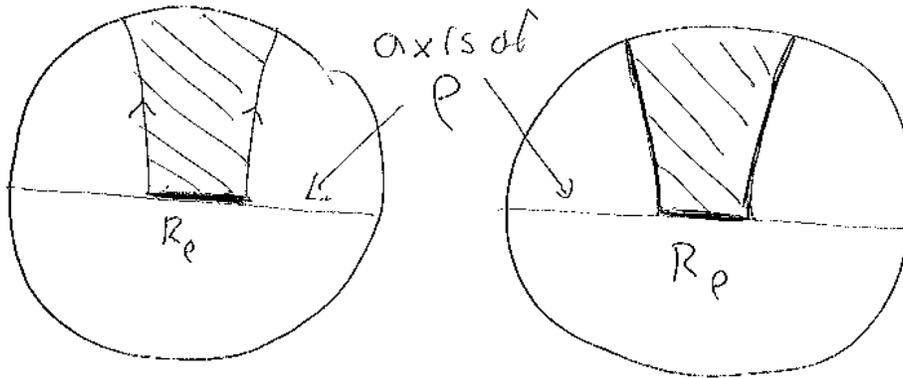


Figure 4: Possible quotients for the centralizer of a reflection. The left shows a funnel, the right shows a planar domain.

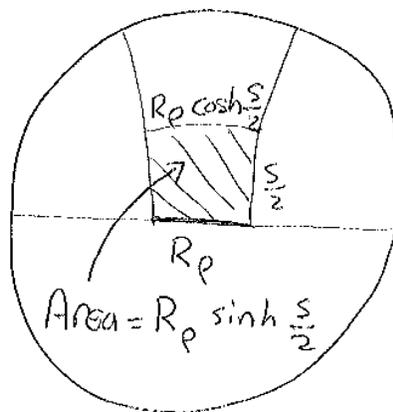


Figure 5: The contribution of a reflection to the counting trace.

groups Γ_j gives $R \sinh \frac{s}{2}$. \square

Now we turn to conjugacy classes of rotations. Each such is associated to a unique conepoint or boundary corner. From a conepoint of order n we get $n - 1$ rotations through angles $\frac{2\pi k}{n}$, $k = 1, \dots, n - 1$. For each such rotation γ the centralizer $Z(\gamma, \Gamma_j)$ is cyclic of order n , consisting of just these $n - 1$ rotations, together with the identity. The quotient $Z(\gamma, \Gamma_j) \backslash H^2$ is an infinite cone of cone angle $\frac{2\pi}{n}$. For a boundary corner of angle $\frac{\pi}{n}$ the centralizer is a dihedral group of order $2n$, and the quotient is an infinite sector of angle $\frac{\pi}{n}$. This quotient is half of a cone of angle $\frac{\pi}{n}$, and it should be clear that the contribution to $C(s)$ of rotations associated to a boundary corner is just half what it is for a conepoint—which is why a boundary corner counts as half a cone-point. So we can forget about boundary corners, and consider just conepoints.

Denote by $g_n(s)$ the total contribution to the counting trace of a conepoint of order n . This contribution is the sum of the contributions of the $n - 1$ non-trivial conjugacy classes of rotations associated to the conepoint. Denote the contribution of the conjugacy class of the rotation through angle $\frac{2\pi k}{n}$ by $g_{k,n}(s)$.

Lemma 2. *The total contribution of a conepoint of order n to the counting trace is $g_n(s) = \sum_{k=1}^n g_{k,n}(s)$, where*

$$g_{k,n}(s) = \frac{1}{n} f_{\frac{2\pi k}{n}}(s),$$

where

$$f_{\theta}(s) = 2\pi \left(\sqrt{1 + \frac{\sinh^2 \frac{s}{2}}{\sin^2 \frac{\theta}{2}}} - 1 \right).$$

Proof. It should be clear that

$$g_{k,n}(s) = \frac{1}{n} f_{\frac{2\pi k}{n}}(s),$$

where $f_{\theta}(s)$ is the area $2\pi(\cosh r - 1)$ of a hyperbolic disk of such a radius r that a chord of length s subtends angle θ . (A fundamental domain for the cone hits a fraction $\frac{1}{n}$ of any disk about the center of rotation.) We just have to make sure that we have the correct formula for f_{θ} . The quantities r, θ, s satisfy

$$\sinh r = \frac{\sinh \frac{s}{2}}{\sin \frac{\theta}{2}}.$$

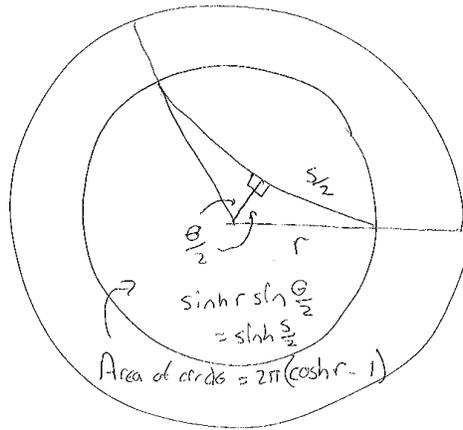


Figure 6: The circle whose chord of length s subtends angle θ

(See Figure 6.) Thus

$$\begin{aligned}
f_\theta(s) &= 2\pi(\cosh r - 1) \\
&= 2\pi\left(\sqrt{1 + \sinh^2 r} - 1\right) \\
&= 2\pi\left(\sqrt{1 + \frac{\sinh^2 \frac{s}{2}}{\sin^2 \frac{\theta}{2}}} - 1\right). \quad \square
\end{aligned}$$

Hopefully it will seem absolutely incredible that there might be any non-trivial linear relation between these functions f_θ . There isn't:

Lemma 3. *The functions $f_\theta, 0 < \theta \leq \pi$ are linearly independent: No non-trivial linear combination of any finite subset of these functions vanishes identically on any interval $0 < s < S$.*

Proof. It suffices to show linear independence of the family of functions $\sqrt{1 + au} - 1, 1 < a < \infty$, on all intervals $0 < u < U$. (Set $u = \sinh^2 \frac{t}{2}$ and $a = \frac{1}{\sin^2 \frac{\theta}{2}}$.) Actually, functions of this form remain independent when a is allowed to range throughout the punctured complex plane $\mathbf{C} - \{0\}$, and not just over the specified real interval. A quick way to see this is to write in Taylor series

$$\begin{aligned}
g(u) &= \sqrt{1 + u} - 1 \\
&= \frac{1}{1!} \binom{1}{\frac{1}{2}} u + \frac{1}{2!} \binom{1}{\frac{1}{2}} \binom{-1}{\frac{1}{2}} u^2 + \frac{1}{3!} \binom{1}{\frac{1}{2}} \binom{-1}{\frac{1}{2}} \binom{-3}{\frac{1}{2}} u^3 + \dots \\
&= b_1 u + \frac{b_2}{2} u^2 + \frac{b_3}{6} u^3 + \dots,
\end{aligned}$$

where all we will need is that

$$b_1, b_2, \dots \neq 0.$$

If a linear combination $\sum_{i=1}^n c_i g(a_i u)$ is to vanish on $0 < u < U$, then all its derivatives must vanish at $u = 0$:

$$\sum_{i=1}^n c_i a_i^k b_k = 0, \quad k = 1, 2, \dots$$

(We ignore the constant term because all the functions involved here vanish at $u = 0$.) Dividing by b_k gives

$$\sum_{i=1}^n c_i a_i^k = 0, \quad k = 1, 2, \dots$$

Could this system of equations for c_1, \dots, c_n have a non-trivial solution? If so, then the subsystem consisting of only the first n of these equations would have a non-trivial solution:

$$\sum_{i=1}^n c_i a_i^k = 0, \quad k = 1, \dots, n.$$

Let $d_i = c_i a_i$, so that

$$\sum_{i=1}^n d_i a_i^{k-1} = 0, \quad k = 1, \dots, n.$$

The n -by- n matrix $(a_i^{k-1})_{1 \leq i, j \leq n}$ of this system of linear equations is a Vandermonde matrix, with determinant $\prod_{i < j} (a_i - a_j)$, so as long as the a_i are distinct the system has no non-trivial solution. \square

Note. In appealing to the fact that none of the derivatives of $\sqrt{1+u}$ happen to vanish at $u = 0$, we are seizing upon an accidental feature of the problem. A more robust proof can be based on the following lemma.

Lemma 4. *Let $f(z) = \sum c_k x^k$ be analytic at 0. Suppose $g(z) = \sum_{i=1}^n a_i f(\alpha_i z)$ with all $|\alpha_i| < 1$. If the Taylor series for f and g agree then f is a polynomial.*

Proof. We must have

$$\sum_{i=1}^n a_i \alpha_i^k = 1$$

for any k for which $c_k \neq 0$. If f is not a polynomial, we can choose k big enough so that the left side is small, and arrive at a contradiction. \square

Note. It would be interesting to know the extent to which this result still holds without the requirement that $a_i < 1$. It will suffice to understand the case where all $|\alpha_i| = 1$. As an example, if we take

$$f(z) = \frac{z + z^2}{1 - z^3} = z + z^2 + z^4 + z^5 + z^7 + z^8 + \dots$$

we can write

$$f(z) = -f(\omega z) - f(\omega^2 z)$$

where $\omega = e^{2\pi i/3}$ is a cube root of unity. Are there examples where the α_i are not roots of unity?

Completion of the proof of Theorem 1. Subtract from $C(s)$ from the counting trace the volume $C(0)$ and the contributions of the reflectors. Because of the linear independence of the functions f_θ , we can identify the number of cone-points of highest order. (It should go without saying that this includes the half-cone-points at boundary corners.) Subtract out their combined contribution from $C(s)$. Now we can identify the number of cone-points of the next highest order, and subtract out their contribution. Proceed until all cone-point contributions have been removed. What remains arises from the closed geodesics.

In [5], we showed that if M is a manifold, the characteristics of the closed geodesics are determined by the counting trace, and hence by the spectrum. The reason is that any possible scenario for a counterexample requires the participation of too many geodesics. Roughly speaking, any balance between primitive orientation-reversing and orientation-preserving geodesics gets destroyed when you look at imprimitive iterates of the geodesics, so you have to keep adding new geodesics to compensate. The proof in [5] carries over here essentially without change. \square

7 Implications

Let $G = \text{Isom}(H^2) \simeq PGL(2, \mathbf{R})$ denote the group of isometries of H^2 . Note that G has two connected components, corresponding to orientation-preserving and orientation-reversing isometries. A hyperbolic 2-orbifold M can be written as a union of quotients of H^2 by discrete cocompact subgroups $\Gamma_j \subset G$, one for each connected component of M :

$$M = \cup_j \Gamma_j \backslash H^2.$$

M is naturally covered by

$$\bar{M} = \cup_j \Gamma_j \backslash G.$$

\bar{M} is the bundle of all orthonormal frames of M , not just those that have a particular orientation; every point x of M is covered by two disjoint circles in \bar{M} , corresponding to the two orientation classes of frames of the tangent space $T_x M$. Note that there will be two connected components of \bar{M} for each orientable component of M , and one for each non-orientable component, because dragging a frame around an orientation-reversing loop (e.g. one that simply bumps off a mirror boundary) will take you from one orientation class of frames to the other.

G acts naturally on the right on \bar{M} , and hence on $L^2(\bar{M})$. This linear representation of G is analogous to the matrix representation of a finite permutation group. A finite permutation representation ρ is determined up to linear equivalence by its character χ , with $\chi(g) = \text{tr}\rho(g)$ counting the fixed points of g . The situation here should be exactly analogous, the only question being exactly how to define the character. The answer comes from the Selberg trace formula.

For now let's extend the discussion to an arbitrary unimodular Lie group G , possibly disconnected. Let $\Gamma < G$ be a discrete subgroup with compact quotient $\bar{M} = \Gamma \backslash G$. (Notice that we make no mention here of M , though in the intended application \bar{M} will arise from a homogeneous quotient $M = \Gamma \backslash G/K$.) Denote by $\text{Cl}(g, G)$ the conjugacy class of g in G , and by $Z(g, G)$ the centralizer of g in G . Introduce Haar measure ρ^g on $Z_G(g)$, normalized in a consistent (i.e., G -equivariant) way. Attribute to $\text{Cl}(\gamma, \Gamma)$ the weight $\rho^\gamma(Z(\gamma, \Gamma) \backslash Z(\gamma, G))$, and define the character $\chi_{\bar{M}}$ to be the function associating to $g \in G$ the total weight of all conjugacy classes $\text{Cl}(\gamma, \Gamma)$ for which $\text{Cl}(\gamma, G) = \text{Cl}(g, G)$. Extend the definition of $\chi_{\bar{M}}$ to $\bar{M} = \cup_j \Gamma_j \backslash G$ by linearity.

As in the finite case, the character is a *central function*, which means that $\chi_{\bar{M}}(gh) = \chi_{\bar{M}}(hg)$ for all $g, h \in G$, or what is the same, $\chi_{\bar{M}}(g)$ depends only on the conjugacy class $\text{Cl}(g, G)$. In analogy to the finite case, we can think of $\chi_{\bar{M}}(g)$ as measuring (in appropriate units) the size of the fixed point set $\{x \in \bar{M} : xg = x\}$.

Proposition 3 (Berard [1]). *The character $\chi_{\bar{M}}$ determines the representation $L^2(\bar{M})$ up to linear equivalence.*

Proof. Using a version of Selberg's trace formula we can write the trace of the integral operator associated to any smooth function of compact support on G in terms of the values of the character $\chi_{\bar{M}}$. (Cf. Wallach [16, Theorem 2.1], Selberg [14, (2.10) on p. 66].) By standard representation theory, these traces determine the representation. For details, see Bérard [1]. \square

Proposition 4 (DeTurck-Gordon [4]). *The character $\chi_{\bar{M}}$ determines the trace of any natural operator on any natural vector bundle over M .*

Proof. Selberg again. \square

With this preparation, we have the following corollaries of Theorem 1.

Corollary 1 (Character is determined). *The Laplace spectrum of a hyperbolic 2-orbifold M (not necessarily connected) determines the character $\chi_{\bar{M}}$.* \square

Corollary 2 (Representation-equivalence). *Laplace-isospectral hyperbolic 2-orbifolds (not necessarily connected) determine equivalent representations of $\text{Isom}(H^2)$.* \square

Corollary 3 (Strong isospectrality). *If two compact hyperbolic 2-orbifolds (not necessarily connected) are Laplace-isospectral then they have the same spectrum for any natural operator acting on sections of any natural bundle.* \square

8 Counterexamples

The analog of Theorem 1 fails for flat 2-orbifolds. In the connected case it does go through more or less by accident, just because there aren't many connected flat 2-orbifolds. But there are counterexamples among disconnected flat 2-orbifolds. We described such examples briefly in [5].

Here's our premier example, which we call *the $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ example*. Let H_1 denote the standard hexagonal flat torus $\langle (1, 0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle \backslash \mathbf{R}^2$. H_1 has as 2-, 3-, and 6-fold quotients a 2222 orbifold H_2 (this is a regular tetrahedron); a 333 orbifold H_3 ; and a 236 orbifold H_6 . (See Figures 7 and 8.) Spectrally,

$$H_2 + H_6 \equiv 2H_3, \tag{1}$$

meaning that these spaces have the same Laplace spectrum. (In fact, as we'll discuss below, they are isospectral for the Laplacian on k -forms for $k = 0, 1, 2$.) Of course these two spaces match as to volume ($\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$) and number of components ($1 + 1 = 2$). But cone-points do not match. On the left we have a 2222 and a 236, so five cone-points of order 2; one of order 3; one of order 6. On the right we have two 333s, so nine cone-points of order 3.

The reason this example is possible is that in the flat case, the contributions of rotations to the counting kernel differ only by a multiplicative constant. Lumping together the contribution of all the rotations associated with a cone-point of order n , we get something proportional to $\frac{n^2-1}{n}$. (This nice simple formula was discovered by Dryden-Gordon-Greenwald [7].) For a general 2-orbifold with variable curvature, it measures the contribution of

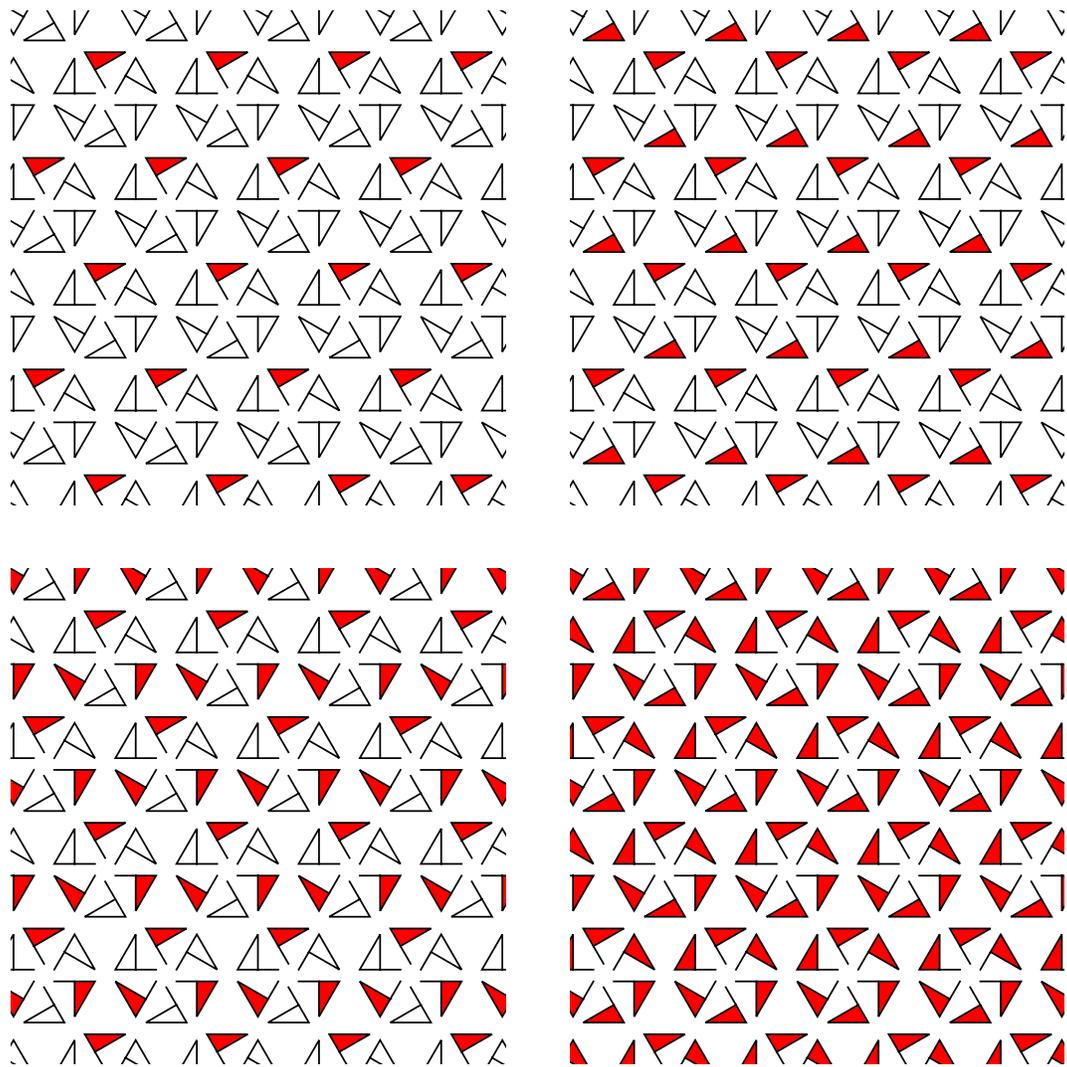


Figure 7: The universal covers of the standard hexagonal torus H_1 and its quotient orbifolds H_2 , H_3 , and H_6 .

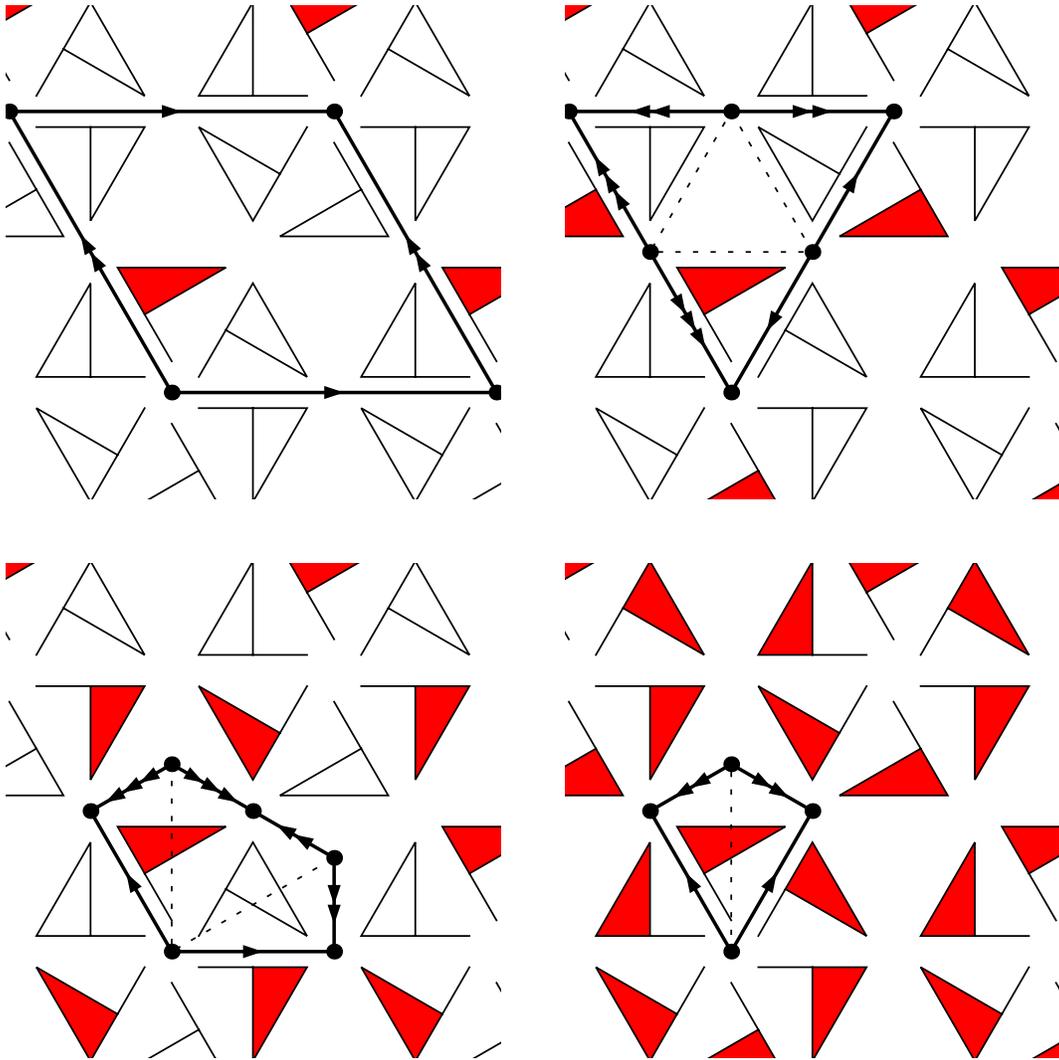


Figure 8: The orbifolds $H_1 = \circ$, $H_2 = 2222$, $H_3 = 333$, and $H_6 = 236$.

conepoints to the short-time asymptotics of the heat trace. In the flat case, the short-time asymptotics determine the entire contribution, because the contributions of all rotations are exactly proportional.

Combining contributions of all conepoints, we get the following totals:

orbifold	contribution	examples
2222 :	$4 \cdot \frac{3}{2} = 6$	H_2, T_2
333 :	$3 \cdot \frac{8}{3} = 8$	H_3
244 :	$\frac{3}{2} + 2 \cdot \frac{15}{4} = 9$	T_4
236 :	$\frac{3}{2} + \frac{8}{3} + \frac{35}{6} = 10$	H_6

The isospectrality $H_2 + H_6 \equiv 2H_3$ arises because $6 + 10 = 2 \cdot 8$ makes conepoint contributions match; we've already observed that the volumes match; and geodesic contributions match because on both sides the covering manifolds are $2H_1$.

We get a second isospectrality

$$H_1 + H_3 + H_6 \equiv 3H_2$$

because $8 + 10 = 3 \cdot 6$, $1 + \frac{1}{3} + \frac{1}{6} = 3 \cdot \frac{1}{2}$, and on both sides the covering manifolds are $3H_1$.

Combining these two relations yields other Laplace-isospectral pairs, e.g.:

$$H_1 + 3H_3 \equiv 4H_2;$$

$$H_1 + 4H_6 \equiv 5H_3;$$

$$2H_1 + 3H_6 \equiv 5H_2.$$

By linearity, all these relations satisfy the condition of having equal volume, number of components, and contributions from conepoints. But look here: We're talking, in effect, about formal combinations of H_1, H_2, H_3, H_6 , so we're in a space of dimension 4. We have three linear conditions, but to our surprise, the subspace they determine has dimension 2. Our three linear conditions are not independent. If we match volume and number of components, agreement of conepoint contributions follows for free. But why? We don't know.

A similar coincidence happens for square tori. Let T_1 denote the standard square torus $\mathbf{Z}^2 \backslash \mathbf{R}^2$. T_1 has as 2-, and 4-fold quotients a 2222 orbifold T_2

and a 244 orbifold T_4 . We're in a 3-dimensional space, so we expect to be out of luck when we impose 3 constraints, but in fact we have the relation

$$T_1 + 2T_4 \equiv 3T_2.$$

We check equality of volume $1 + \frac{2}{4} = \frac{3}{2}$ and number of components $1 + 2 = 3$, and then we find that equality of cone-point contributions follows for free: $2 \cdot 9 = 3 \cdot 6$. Again, why?

These relations apply only to the spectrum of the Laplacian on functions, not 1-forms. When we apply the Selberg formula to 1-forms, there is still interference between spectral contributions of cone-points, which allows our $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ example to continue to slip through: It is isospectral for 1-forms as well as for functions. But we get additional constraints, which taken with those coming from the 0-spectrum force there to be the same number of torus components on each side, and this wipes out the other examples. The reason we have to have the same number of torus components is that this is just half the dimension of the space of 'constant' or 'harmonic' 1-forms, meaning those belonging to the 0-eigenspace of the Laplacian. And since we are expecting just one more linear constraint, this must be it. So, no additional mystery here, though when you do the computation it still seems a little surprising.

The isospectrality of the $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ example does not hold for all natural operators. because the left side admits harmonic differentials that the right side does not. Specifically, look at the real part of a (non-holomorphic) quadratic differential, i.e. an expression which in any conformally correct chart with coordinate z takes the form $\Re f(z)dz^2$, where f is an arbitrary (not necessarily holomorphic) complex-valued function. We can rewrite this as

$$g(x, y)(dx^2 - dy^2) + h(x, y)(2dxdy)$$

where g and h are real. Such quadratic differentials are sections of a natural flat bundle, and in a flat coordinate system the Laplacian passes through to act on g and h . Wherever the section doesn't vanish we can set

$$\Re f(z)dz^2 = 0$$

and solve for dz to get an unoriented line element. A harmonic section (i.e. an eigensection belonging to eigenvalue 0) will have g and h constant in any flat coordinate system, and the line elements will be parallel. In the $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ example only the orbifold H_2 admits a globally parallel line element like this, so the spectra for the Laplacian on these sections are different.

9 Higher dimensions

Version 1 of this paper [6] contains, among many other things omitted here, the following conjecture: ‘It seems likely that in any dimension, if two hyperbolic (or spherical) orbifolds are isospectral on k -forms for all k , then they are representation-equivalent.’

Lauret, Miatello, and Rossetti [12] have shown that this is false in the spherical case. They give many examples of lens spaces that are all-forms-isospectral but not representation-equivalent, beginning in dimension 5. (For a simpler approach to these examples, see DeFord and Doyle [3].)

In the hyperbolic case, the question remains open.

As for the flat case, we saw in Section 8 an example of disconnected 2-orbifolds that are all-forms-isospectral but not representation-equivalent. We have been hoping to find higher-dimensional examples that are connected, and perhaps even manifolds and not just orbifolds, but so far we’ve had no success with this.

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