Conway's drum quilts

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Abstract

A 'transplantable pair' is a pair of glueing diagrams that can be used to create pairs of plane domains that are isospectral for the Laplace operator. We present a host of transplantable pairs worked out by John Conway using his theory of quilts.

1 Introduction

A 'transplantable pair' is a pair of glueing diagrams that can be used to create pairs of plane domains or other spaces that are isospectral for the Laplace operator. The pair is metonymically 'transplantable' because isospectrality of the glued spaces can be proven using Peter Buser's transplantation method, as explained by Buser, Conway, et al. [1] and Conway [3].

John Conway produced a host of transplantable pairs by applying his theory of quilts to small projective groups. I helped by watching admiringly. In this paper I have reproduced his catalog of pairs. The sizes of these pairs are 7, 11, 13, 15, and 21. In [1] we presented the sixteen pairs of sizes 7, 13, and 15, which are treelike and thus give planar isospectral domains. We also presented one of the size 21 pairs, here labeled pair 21(7), which yields the 'homophonic' domains shown in Figure 1. Conway [4, p. 249] dubbed these domains 'peacocks rampant and couchant'.

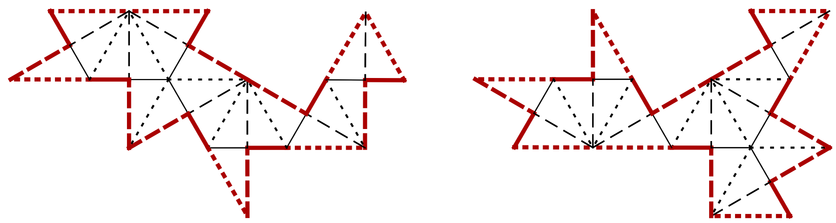


Figure 1: Peacocks rampant and couchant

A transplantable pair can be thought of as arising from a pair of finite permutation actions of the free group on generators a, b, c. These actions are equivalent as linear representations but (if the pair is to be of any use) not as permutation representations. In the examples at hand, each representing permutation is an involution, so these representations factor through the quotient $F = \langle a, b, c : a^2 = b^2 = c^2 = 1 \rangle$, the free product of three copies of the group of order 2.

From any transplantable pair we can get other pairs through the process of braiding, which amounts to precomposing the permutation representations with automorphisms of F, called L and R:

$$L : (a, b, c) \mapsto (aba^{-1}, a, c);$$

$$R : (a, b, c) \mapsto (a, c, cbc^{-1}).$$

Left-braiding a permutation representation ρ can be viewed as first conjugating $\rho(b)$ by $\rho(a)$, i.e. applying the permutation $\rho(a)$ to each index in the cycle representation of $\rho(b)$, and then switching $\rho(a)$ with the new $\rho(b)$. Right-braiding is the same, only with c taking over the role of a.

Pairs of permutations that are equivalent in this way belong to the same *quilt*. We identify pairs that differ only by permuting a, b, c, or by reversing the pair. A quilt has extra structure which we are ignoring here: See Conway and Hsu [5]. This structure makes it easier to understand and enumerate the pairs. But the computer has no trouble churning out all the pairs belonging to the same quilt.

So despite the title and what you might reasonably expect, the only place you will find quilts here is in Appendix A, which reproduces Conway's original quilt calculations.

First we will present the glueing diagrams, and then the corresponding hyperbolic orbifolds.

2 Transplantation diagrams

In the diagrams that follow, the points being permuted are represented by triangles. The permutations corresponding to a, b, c are represented by lines of three styles: dotted, dashed, and solid. Black lines separate pairs of points that are interchanged by the permutation, while red lines (which are also made thicker) indicate fixed points. Red lines in the interior of the diagram separate points that are each fixed, rather than interchanged. Sometimes black lines occur on the boundary of the diagram, which means that the boundary must be glued up. The computer has taken care to lay out the diagrams so that there is at most one pair of thin boundary lines of each type (dotted, dashed, or solid), so that even without the usual glueing arrows there is no ambiguity of how the boundary is to be glued up.

To refer to these pairs, we will write 7(1) for the first pair of quilt 7, $13a(5)$ for the fifth pair of quilt $13a$, etc. This numbering is canonical, given the starting triple (a, b, c), because the quilt has been explored by a 'left-first search'. More canonical, but more cumbersome, are the Conway symbols of the simplest hyperbolic orbifolds that can be obtained from the pair: See Section 3.

For historical reasons, the four quilts of size 11 are called $11f, 11g, 11h, 11i$. The missing quilts (whose diagrams actually have size 12) are to be found in raw form in Appendix A.

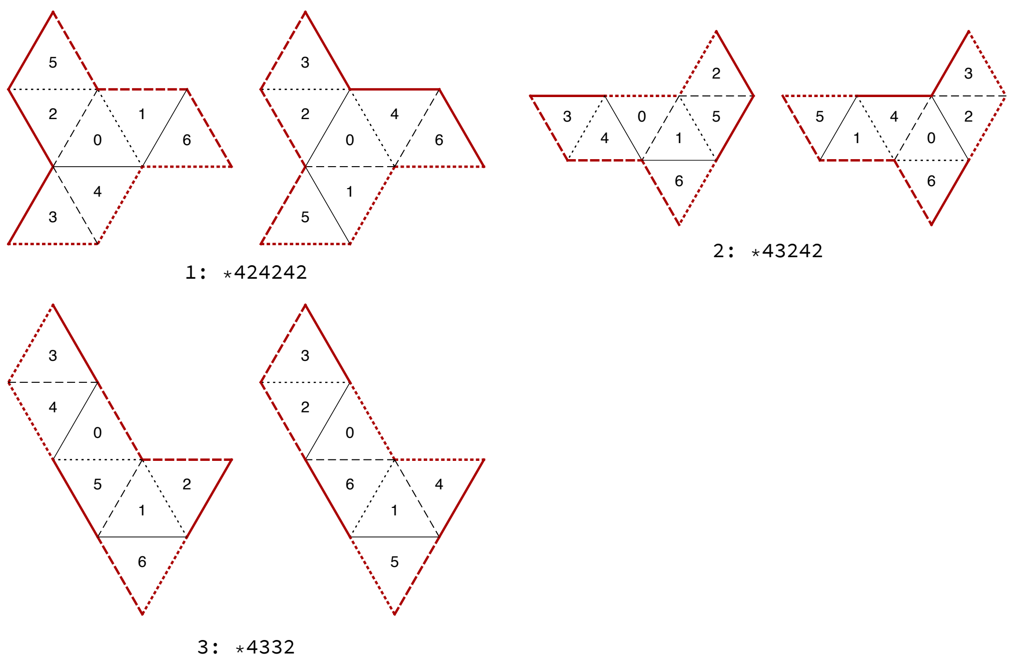


Figure 2: Quilt 7

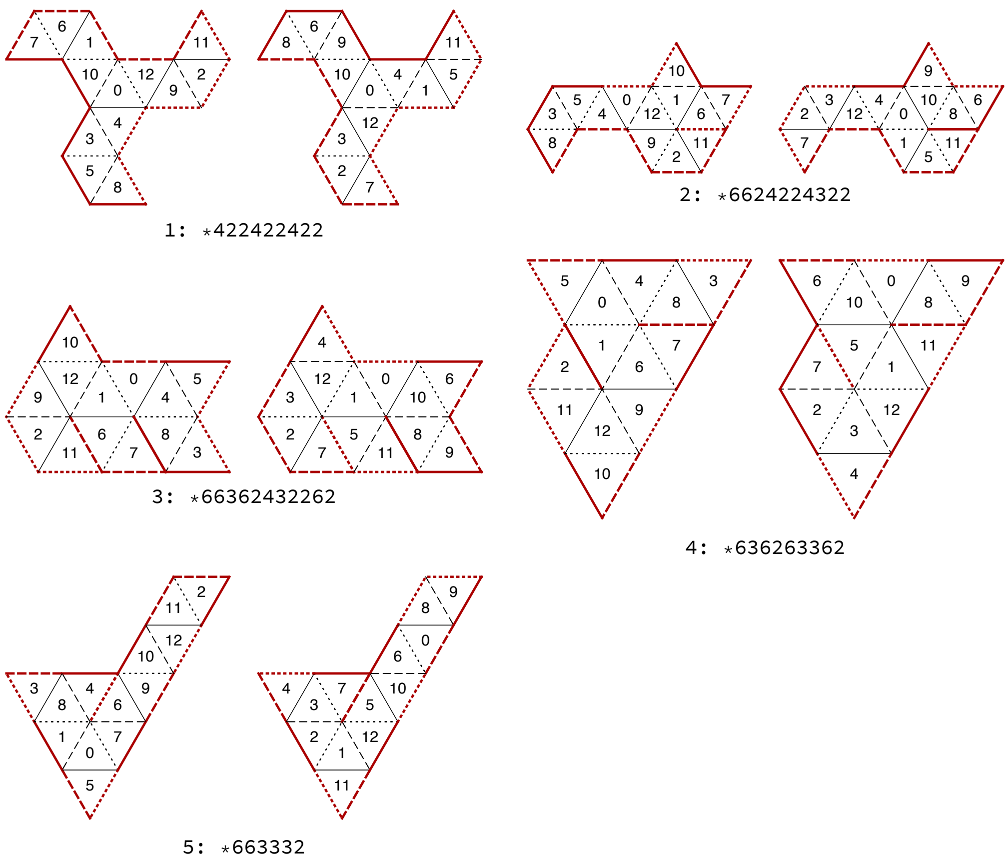


Figure 3: Quilt 13a

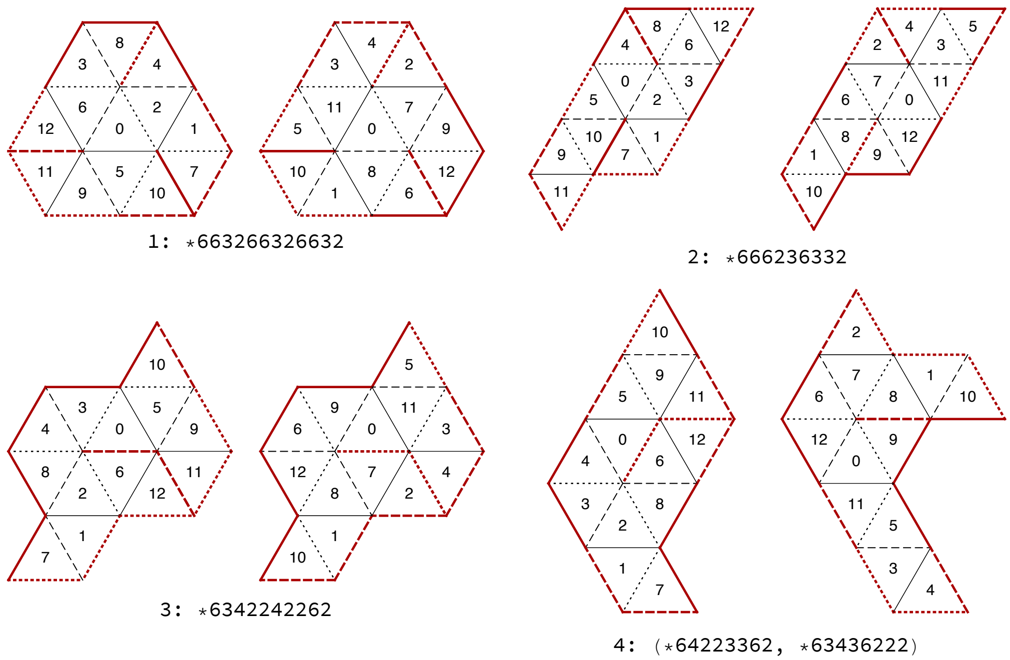


Figure 4: Quilt 13b

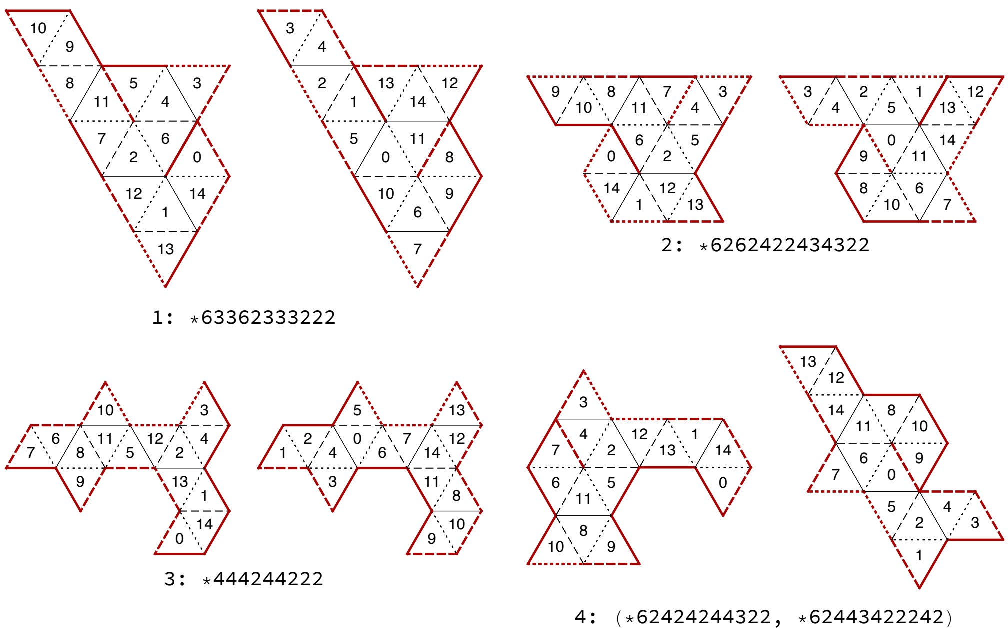


Figure 5: Quilt 15

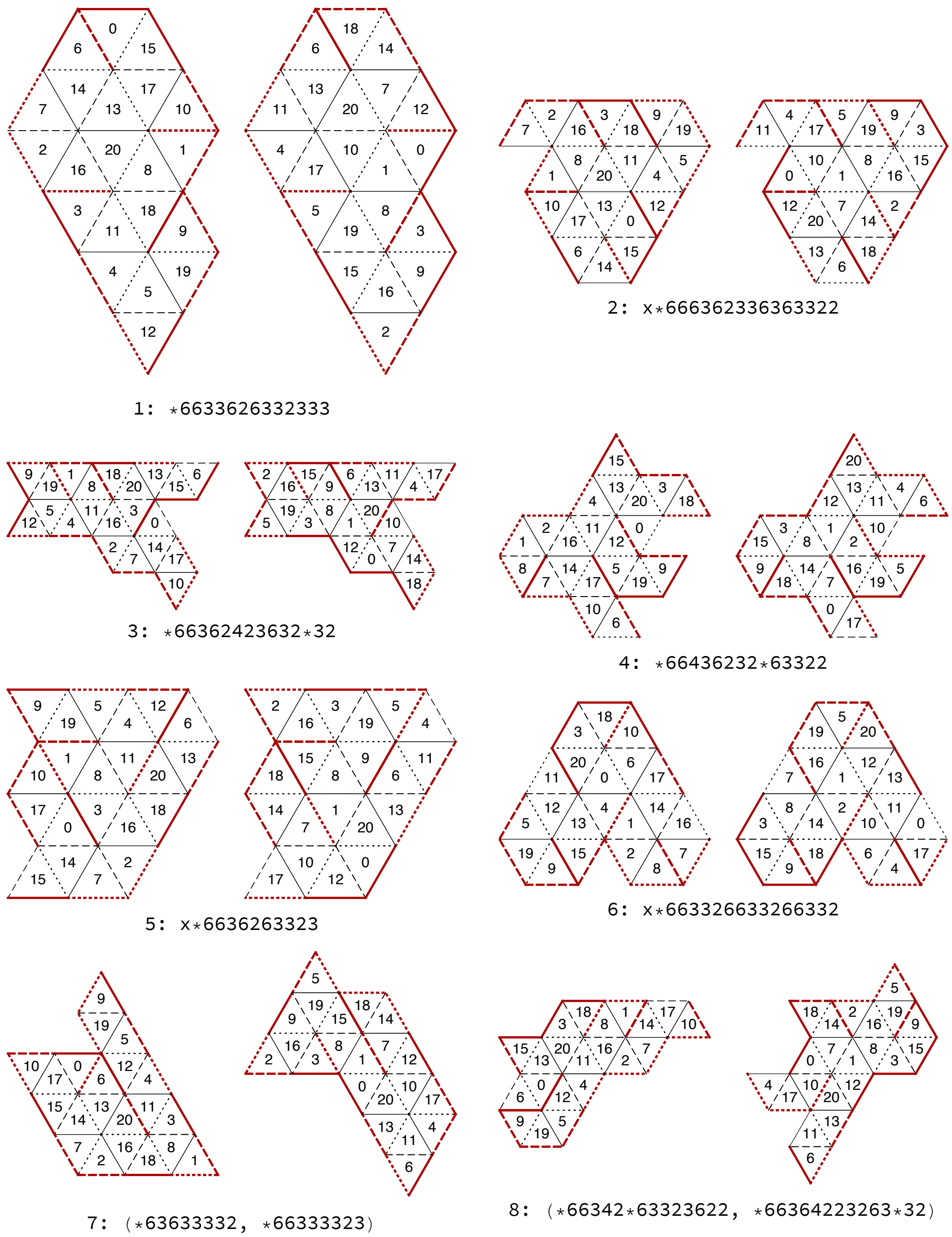


Figure 6: Quilt 21

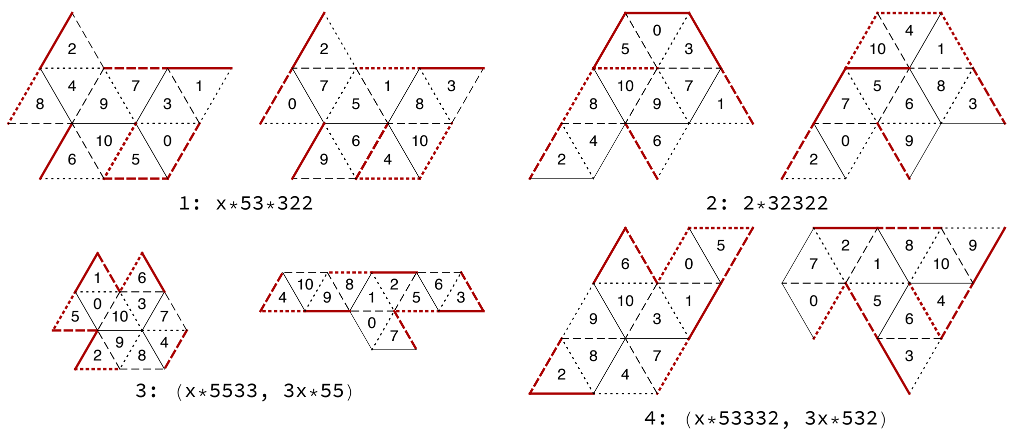


Figure 7: Quilt 11f

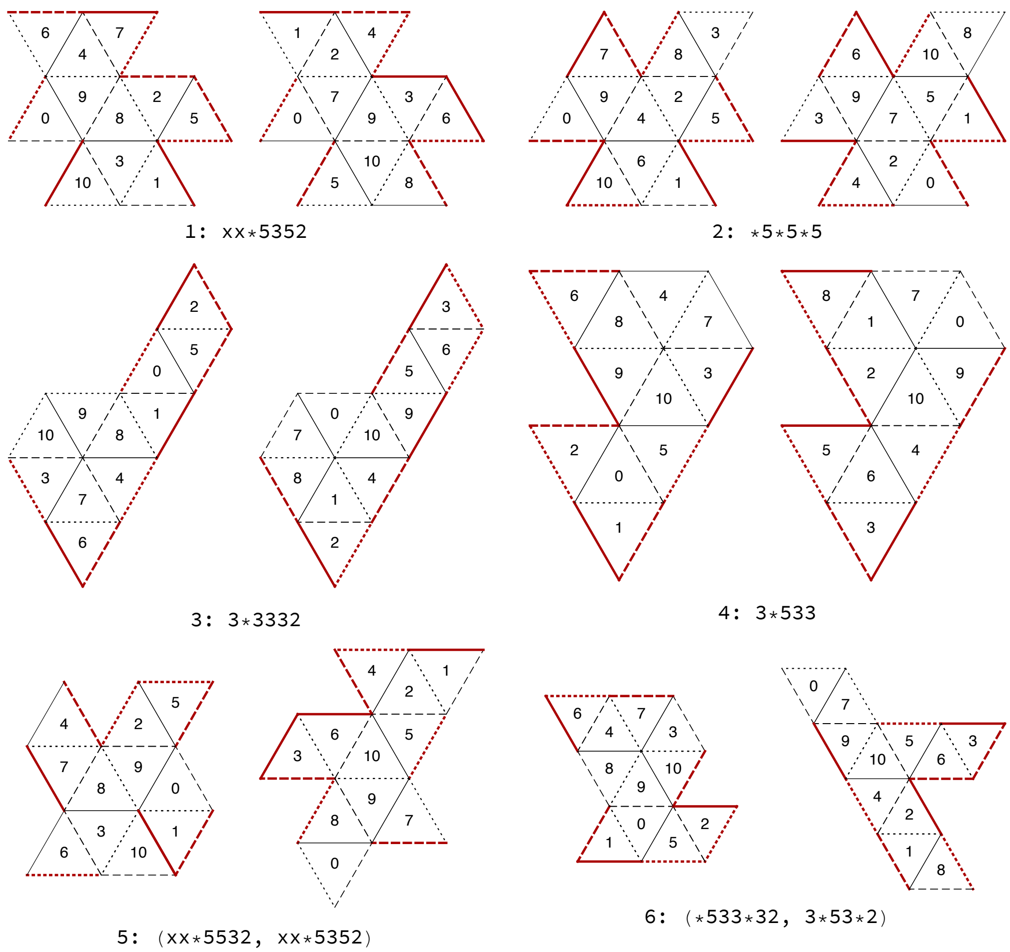


Figure 8: Quilt 11g

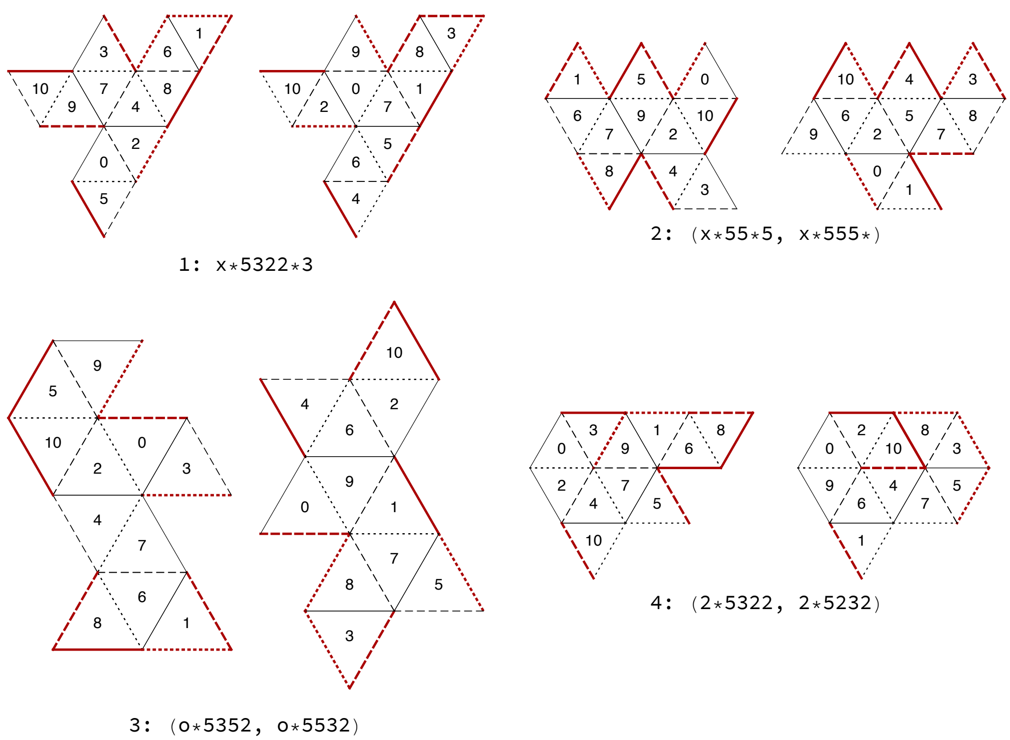


Figure 9: Quilt 11h

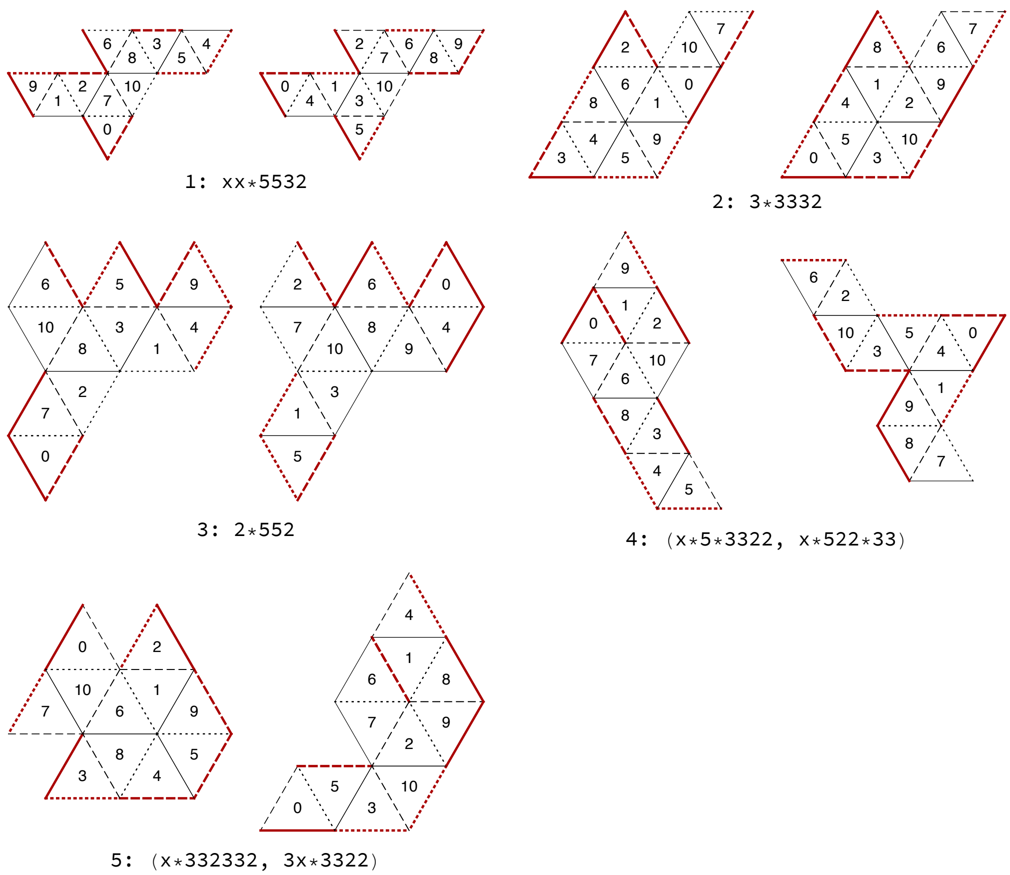


Figure 10: Quilt 11i

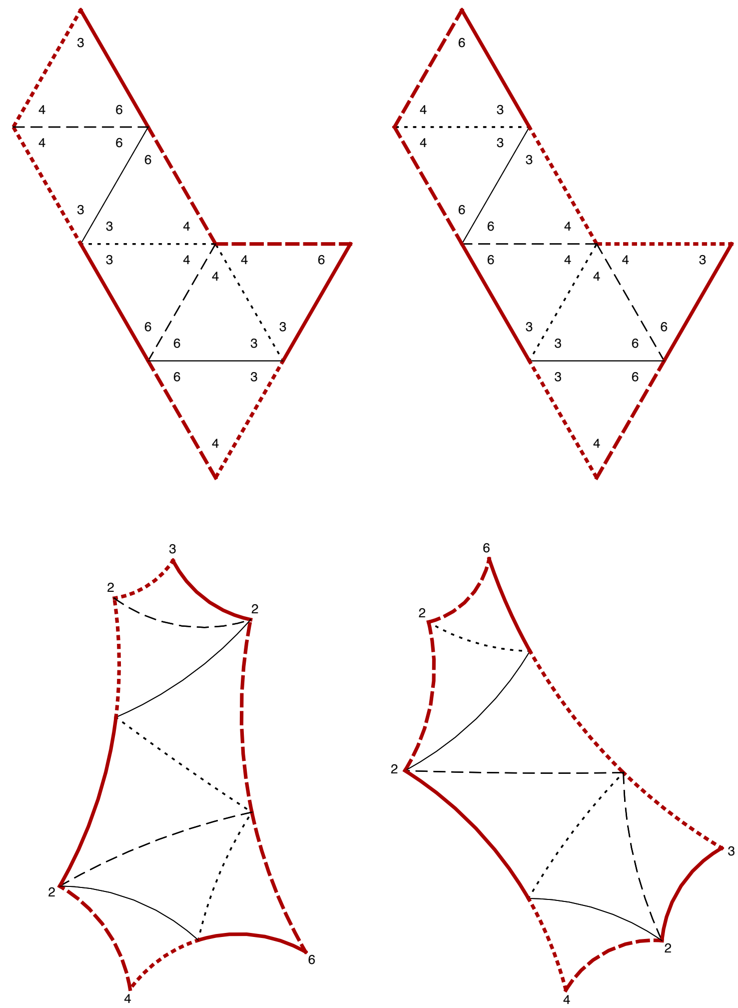


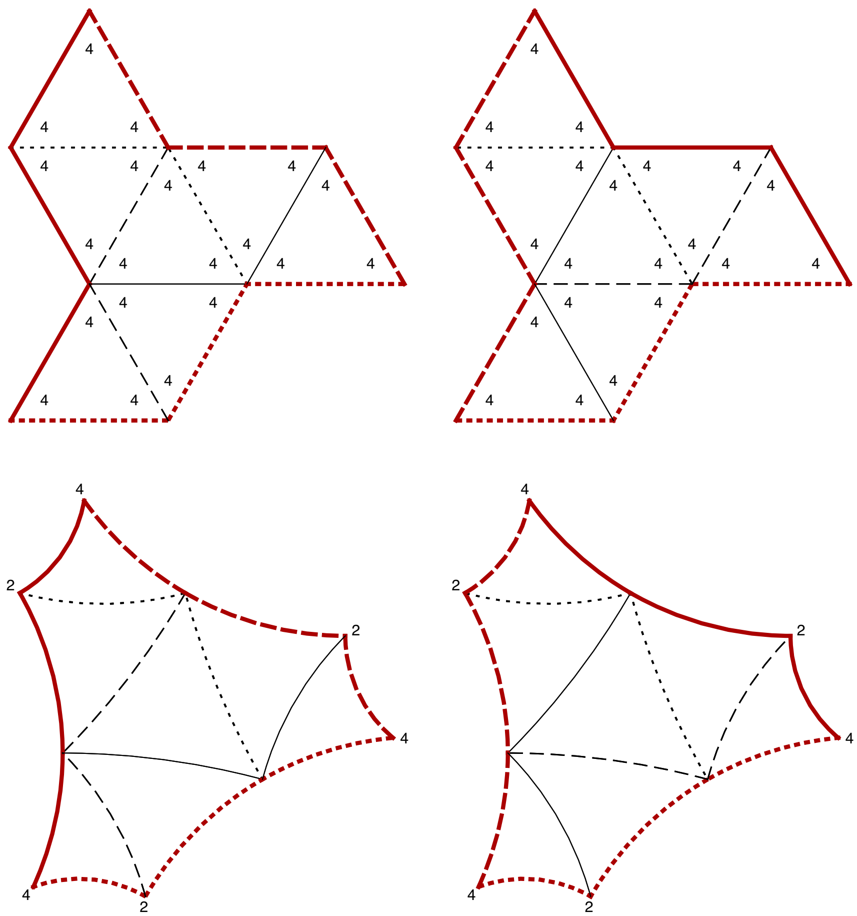
Figure 11: Isospectral hyperbolic hexagons arising from diagram 7(3).

3 Hyperbolic orbifolds

Here are the hyperbolic orbifolds corresponding to these transplantable pairs. These are the simplest hyperbolic orbifolds that can be produced using the glueing data. To get them, for the basic triangle we prescribe angles just small enough to make each interior cone point have cone angle evenly dividing τ, and each boundary corner have angle evenly dividing $\tau/2$. When the members of the pair differ only by permuting the labels a, b, c, the resulting orbifolds are isometric: To get non-isometric pairs we will need to destroy this symmetry by taking one or more of the triangles smaller. Thus, for example, from the pair 7(3) we get the isospectral hexagons shown in Figure 11. This is presumably the simplest pair of isospectral hyperbolic 2-orbifolds.

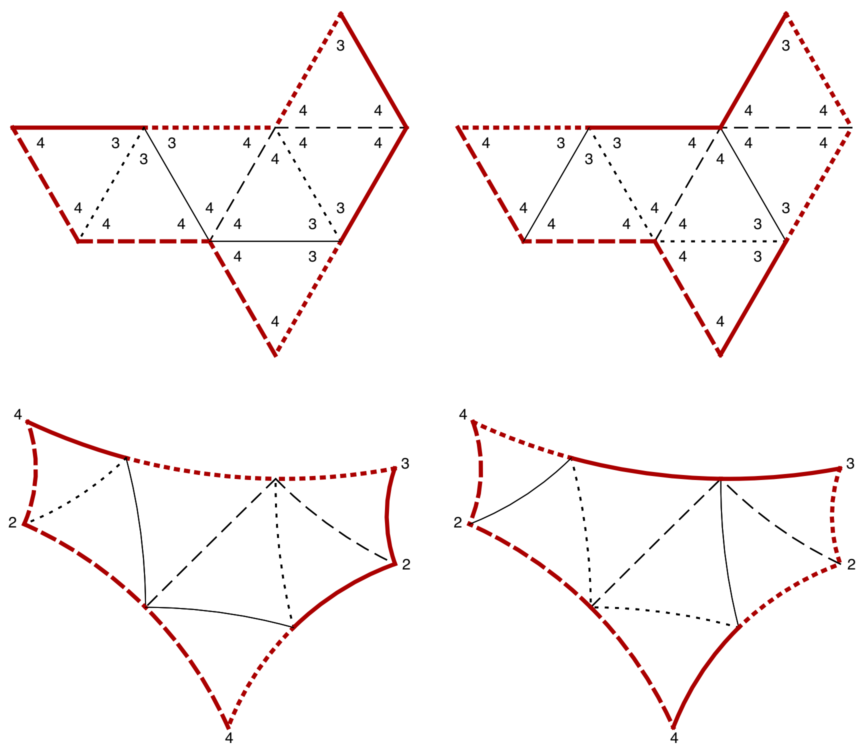
Each figure gives Conway's notation (see Conway [2]) for the associated orbifolds.

Note. The pairs show here are isospectral as hyperbolic 2-orbifolds. We can demote an orbifold to a manifold with boundary, and we will still have isospectrality if we impose Neumann boundary conditions. Dirichlet boundary conditions also work, provided that the manifold with boundary is orientable. This will be the case when the diagrams are treelike, or more generally, when all cycles have even length. If there are cycles of odd length, when we put Dirichlet boundary conditions we must also use twisted functions, i.e. sections of a non-trivial bundle, which change sign when you travel around an orientation-reversing path.



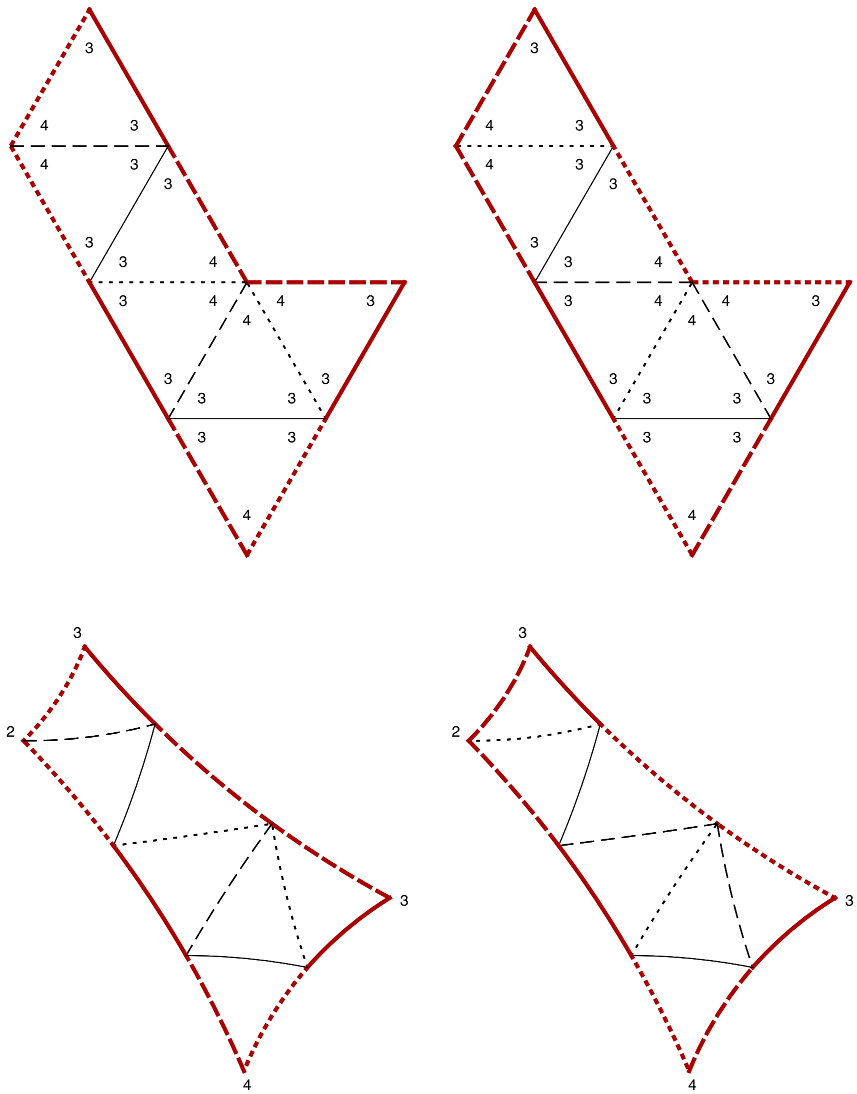
⋆424242

Figure 12: Pair 7(1)



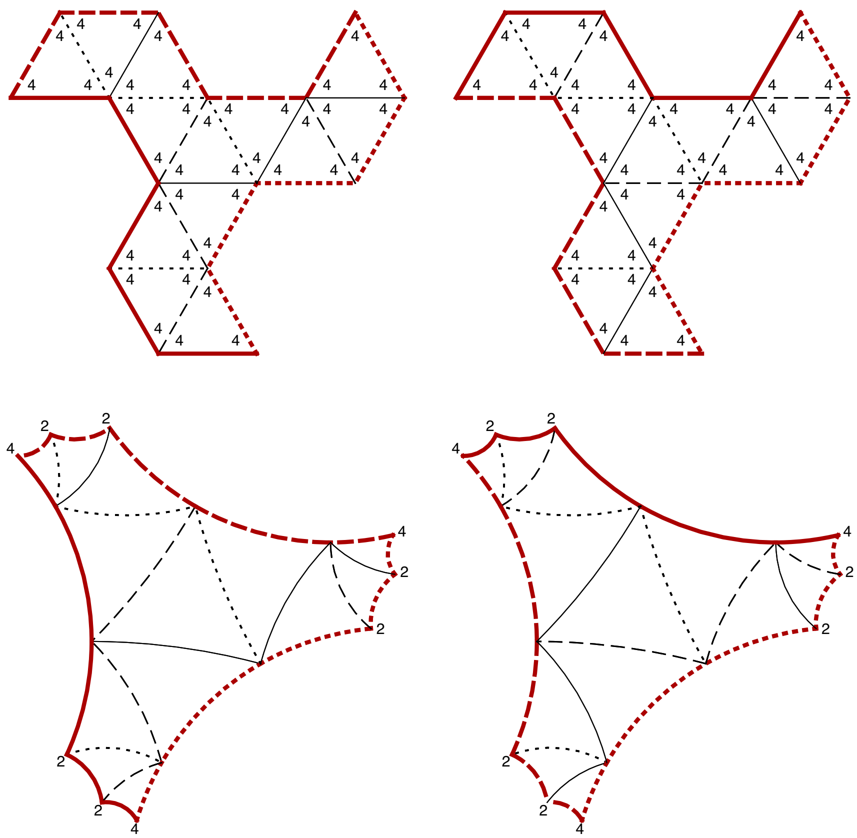
⋆43242

Figure 13: Pair 7(2)



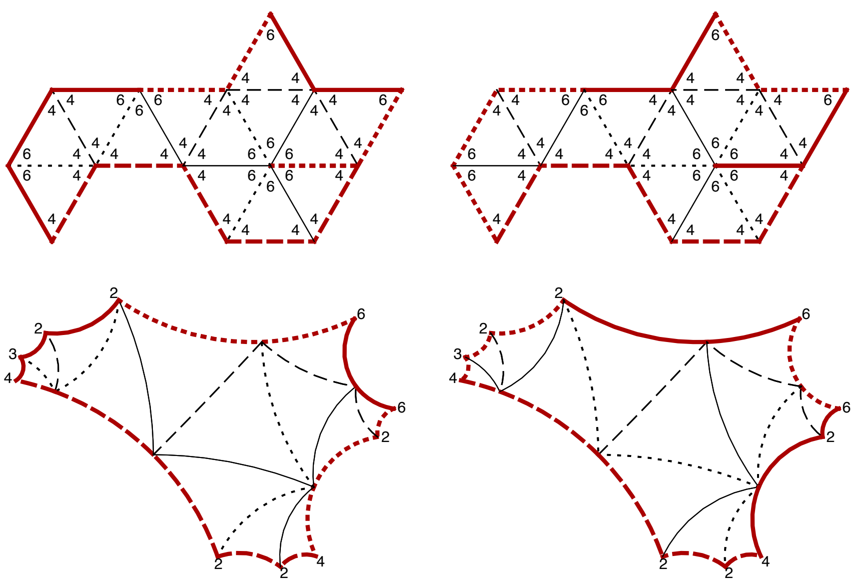
⋆4332

Figure 14: Pair 7(3)



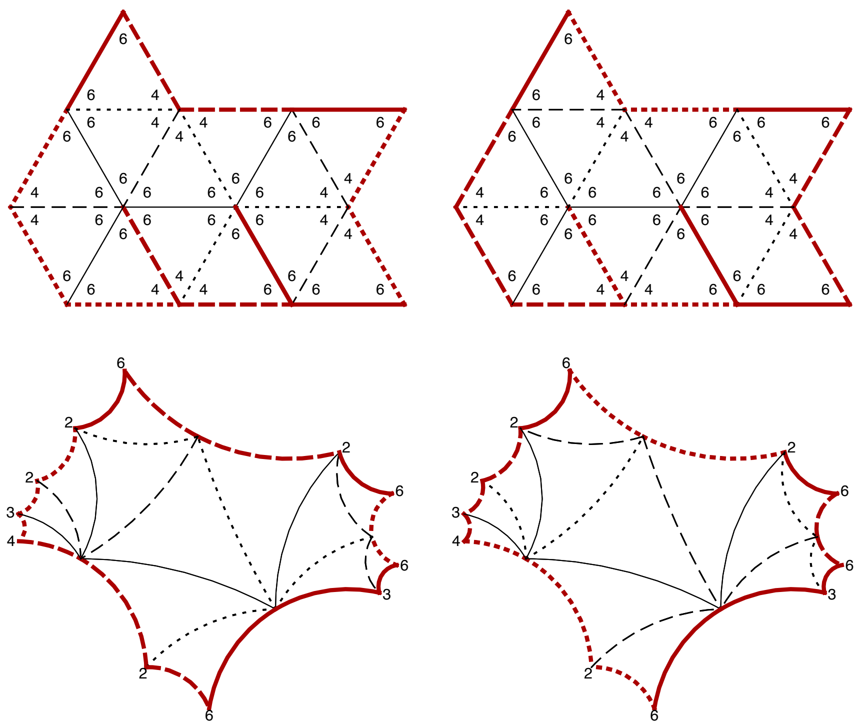
$\star 422422422$

Figure 15: Pair 13a(1)



$\star 6624224322$

Figure 16: Pair 13a(2)



⋆66362432262

Figure 17: Pair 13a(3)

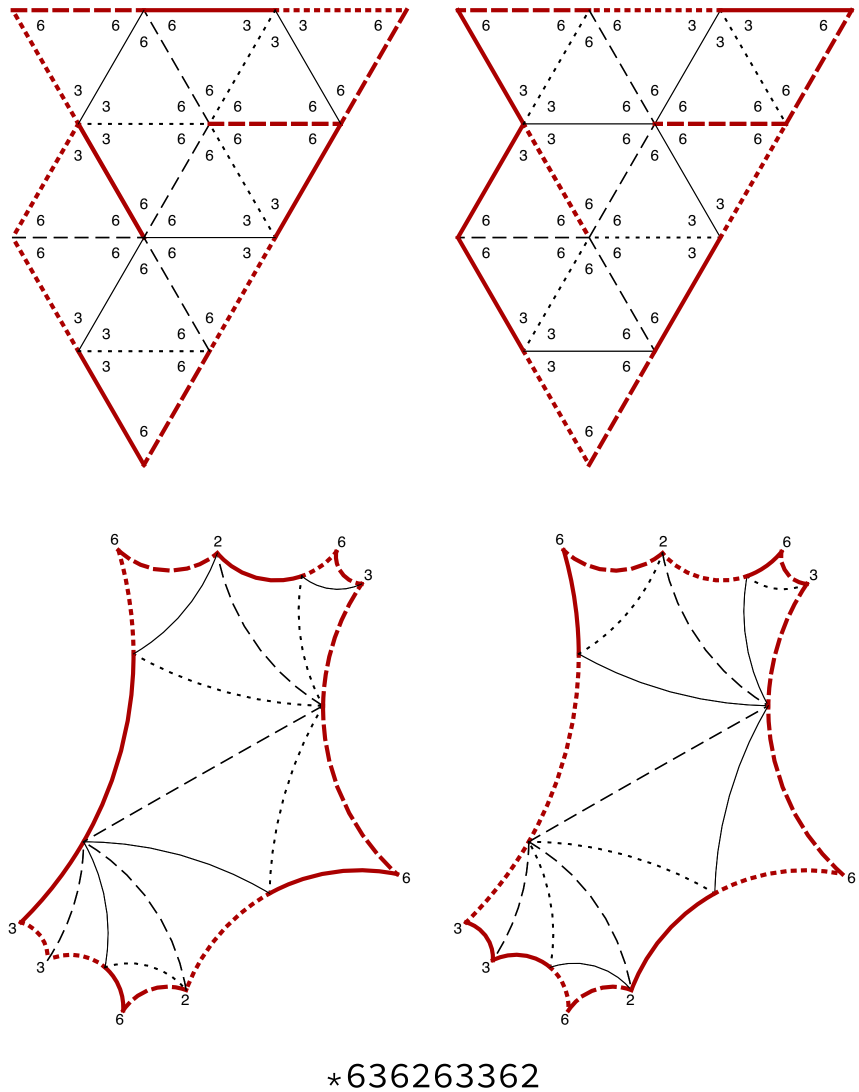
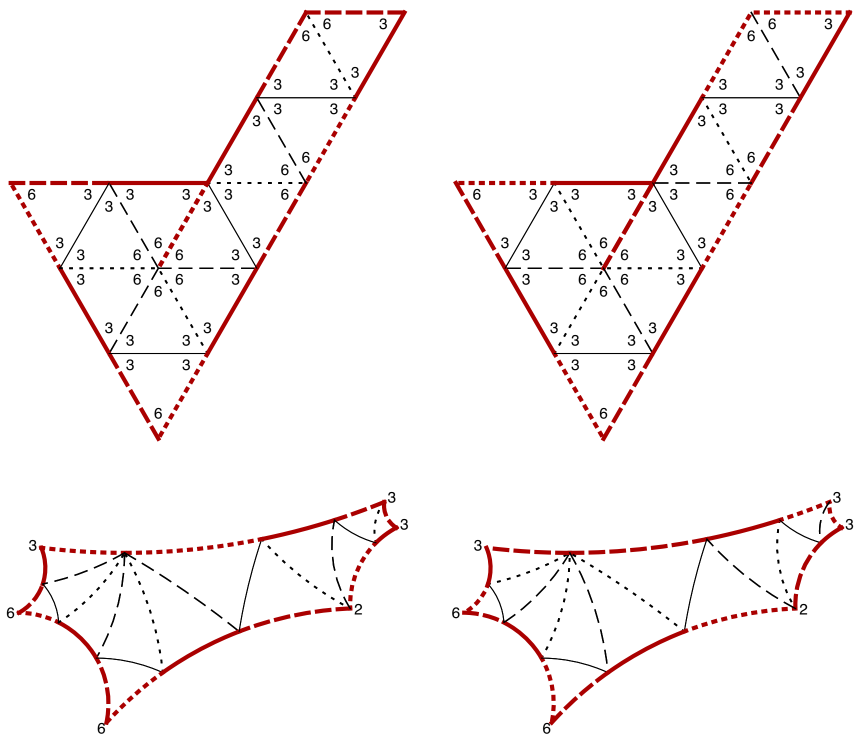
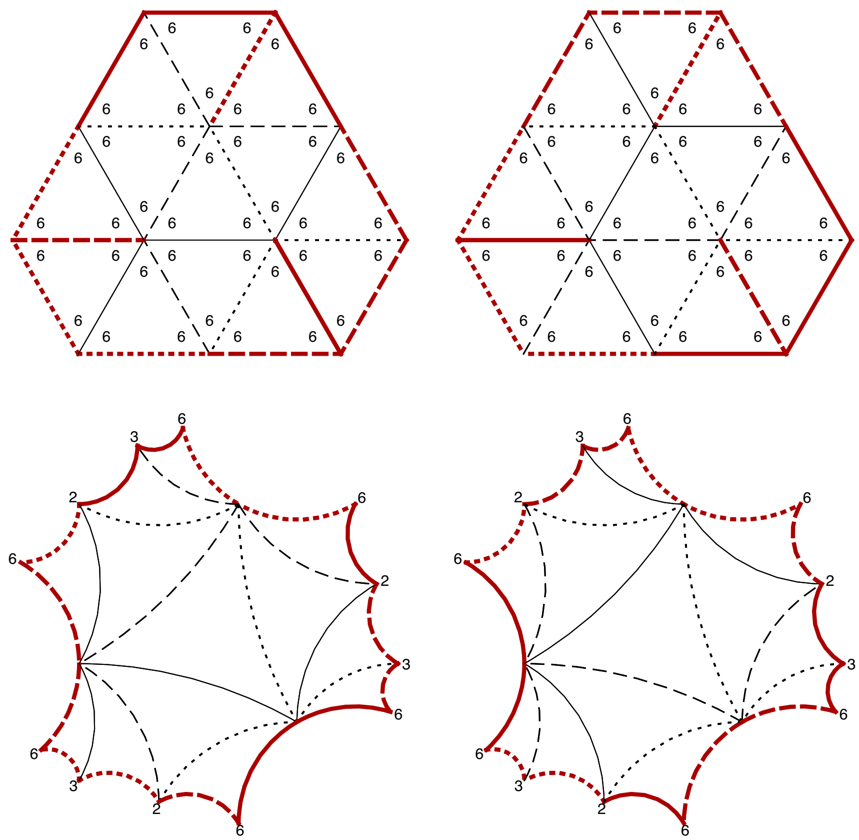


Figure 18: Pair 13a(4)



⋆663332

Figure 19: Pair 13a(5)



$\star 663266326632$

Figure 20: Pair 13b(1)

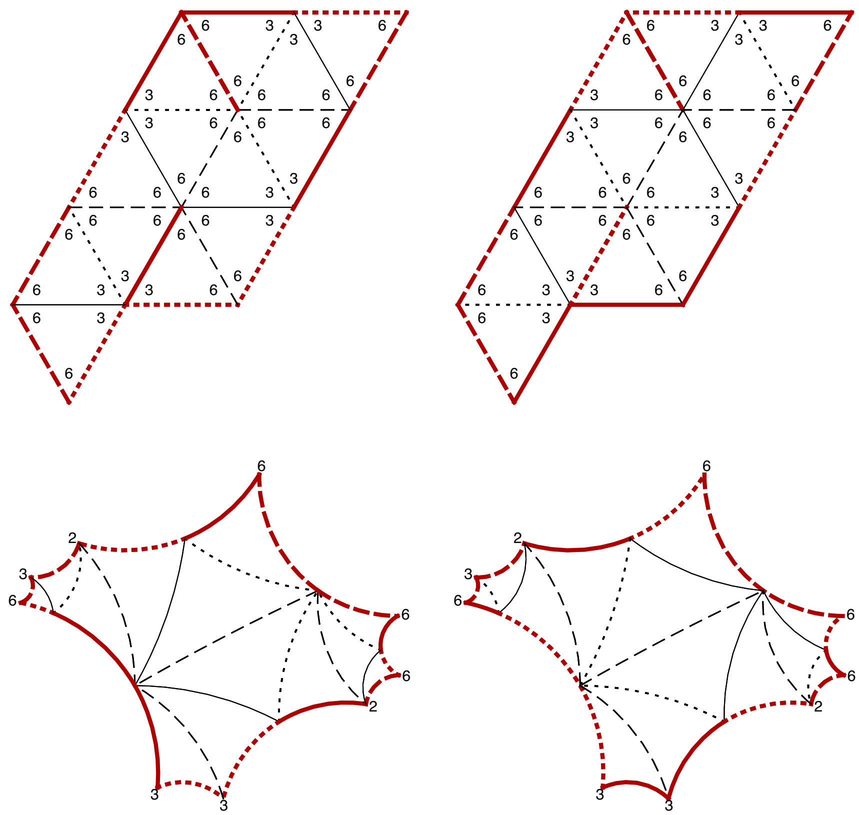
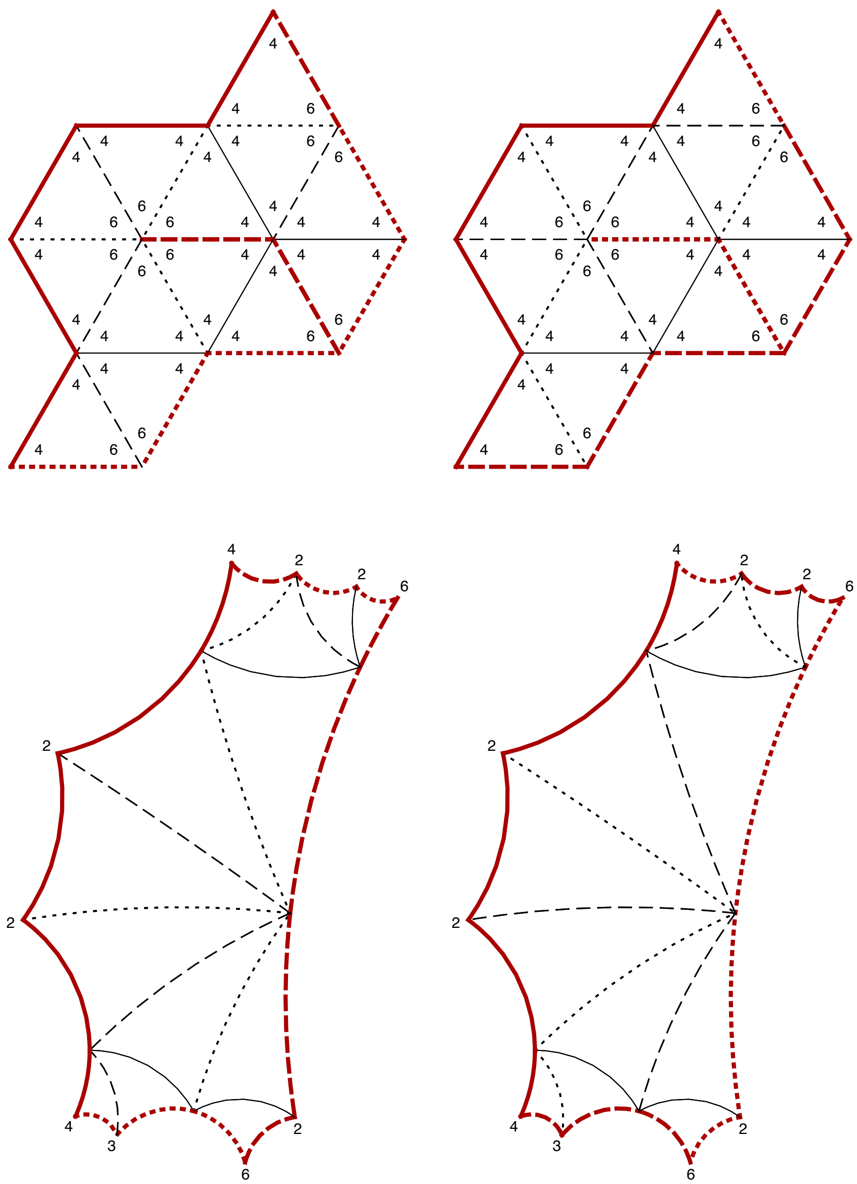
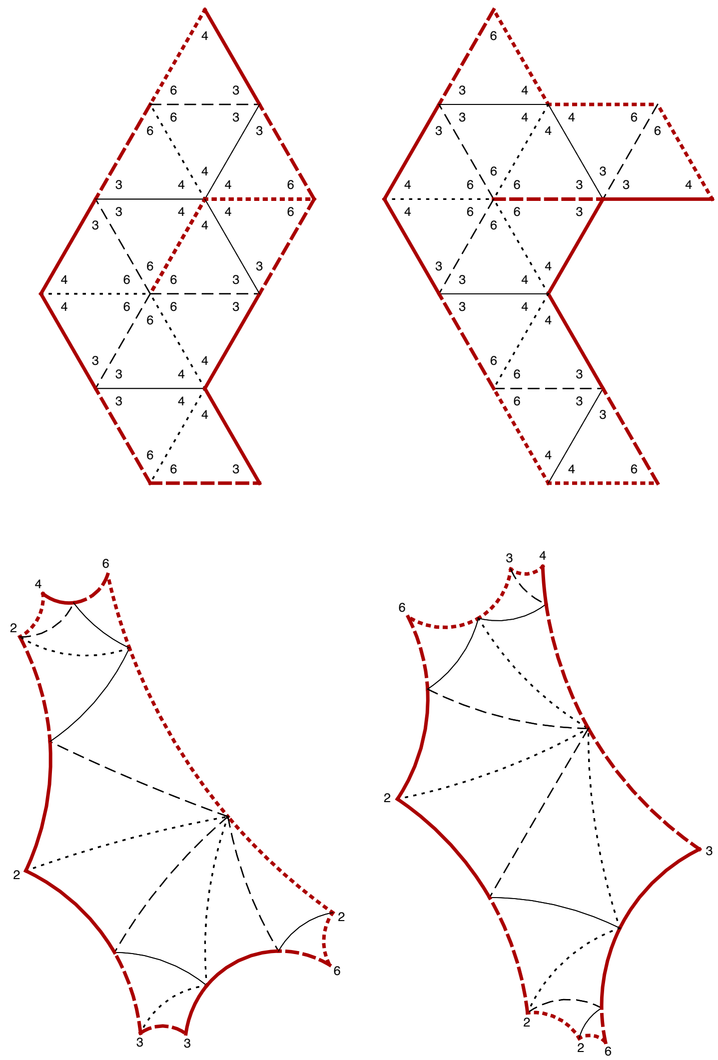


Figure 21: Pair 13b(2)



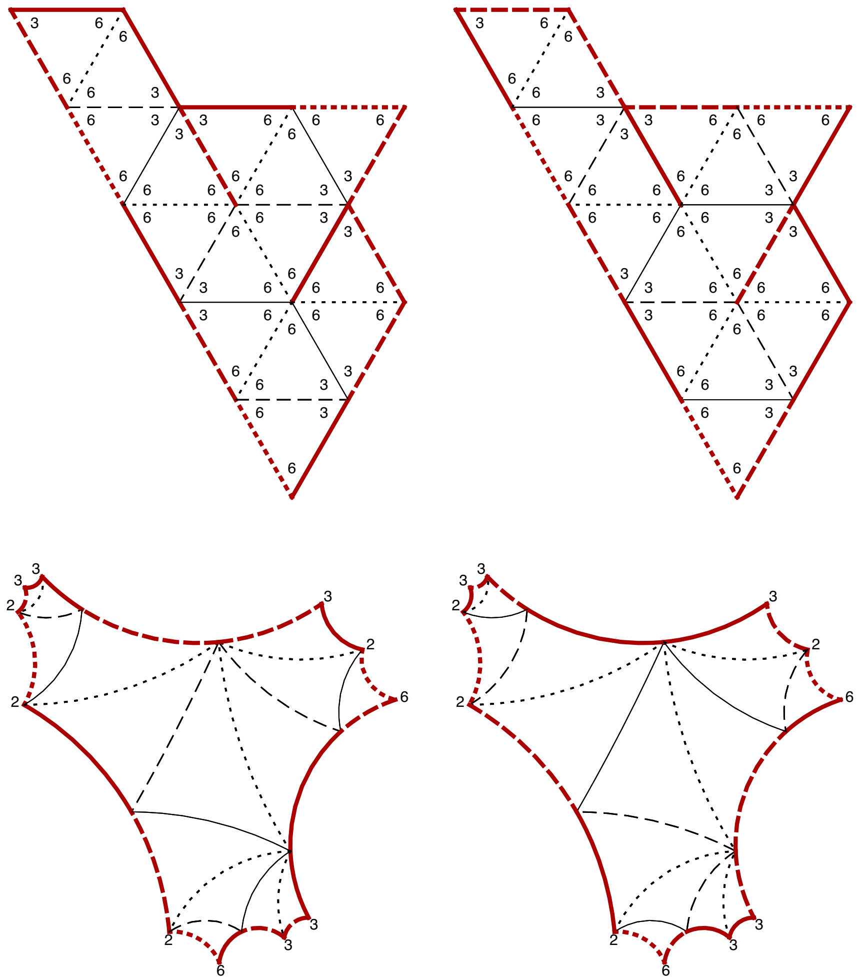
$\star 6342242262$

Figure 22: Pair 13b(3)



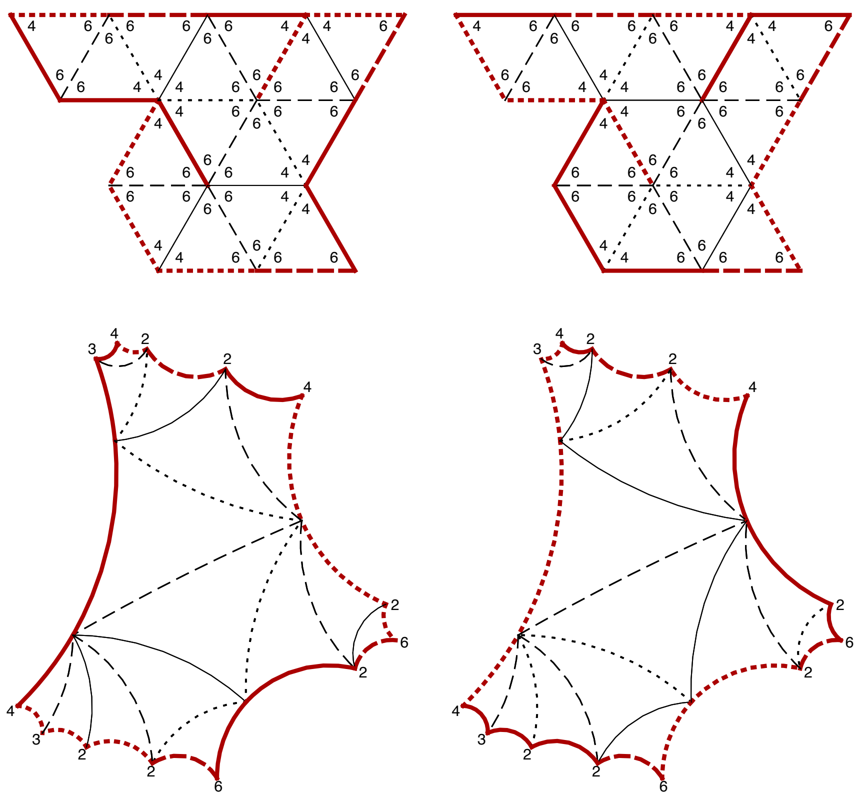
(⋆64223362, ⋆63436222)

Figure 23: Pair 13b(4)



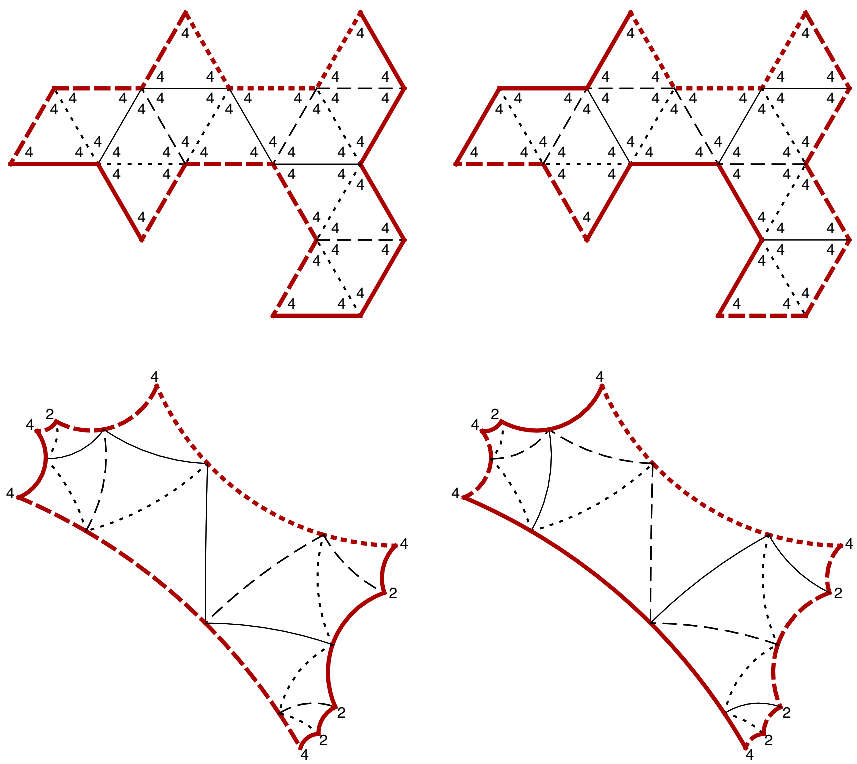
⋆63362333222

Figure 24: Pair 15(1)



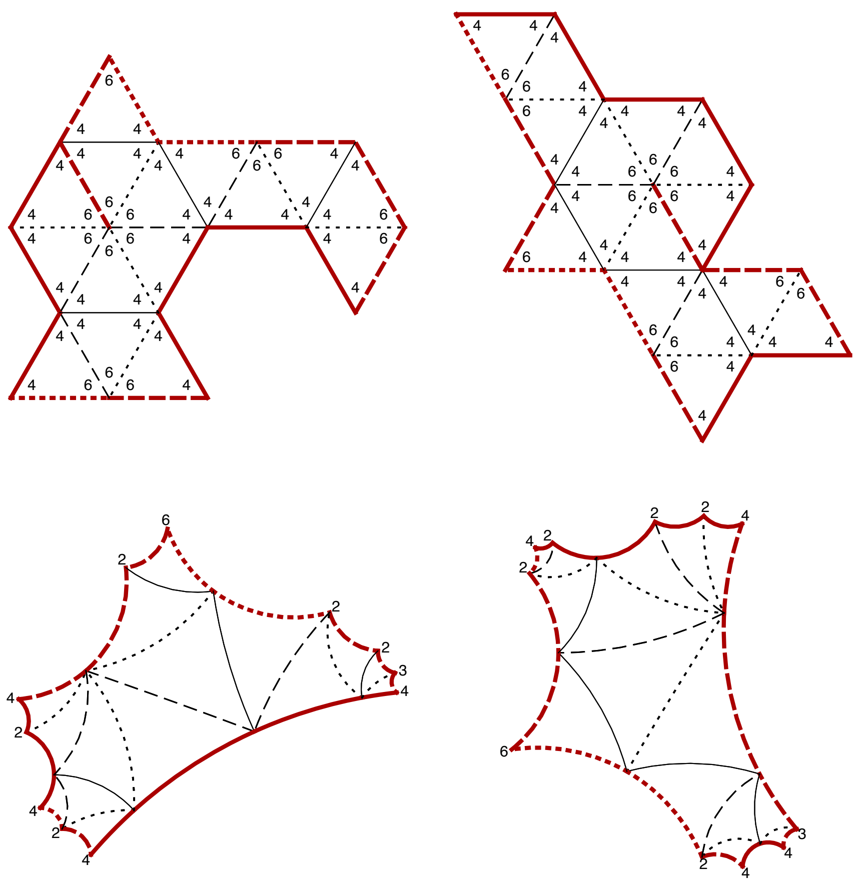
$\star 6262422434322$

Figure 25: Pair 15(2)



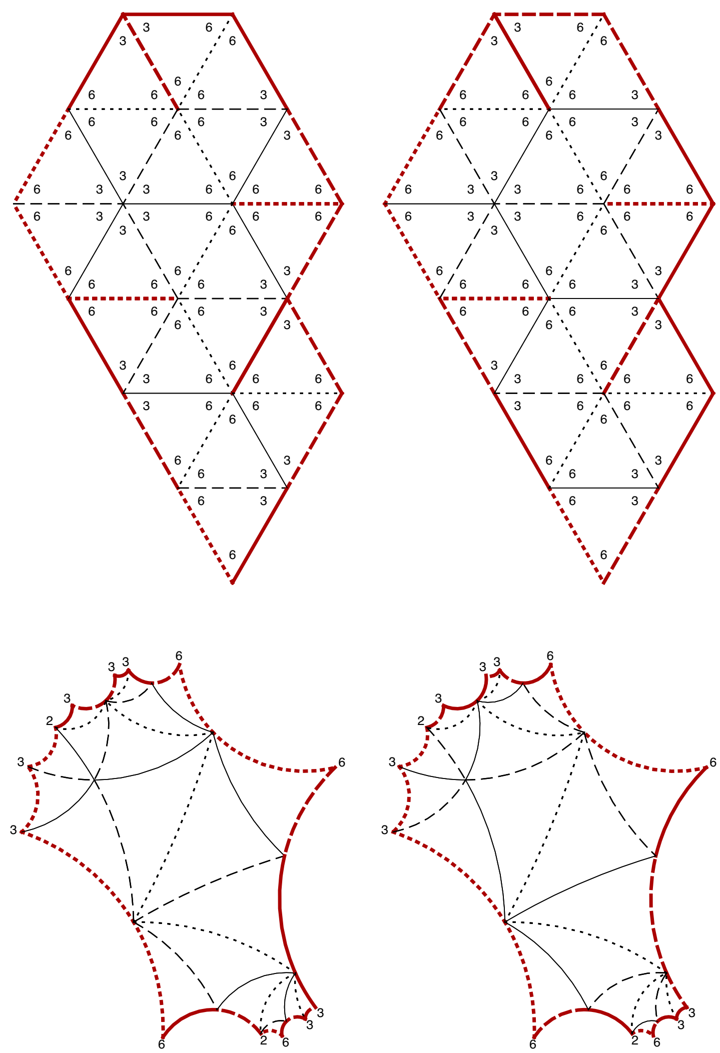
$\star 444244222$

Figure 26: Pair 15(3)



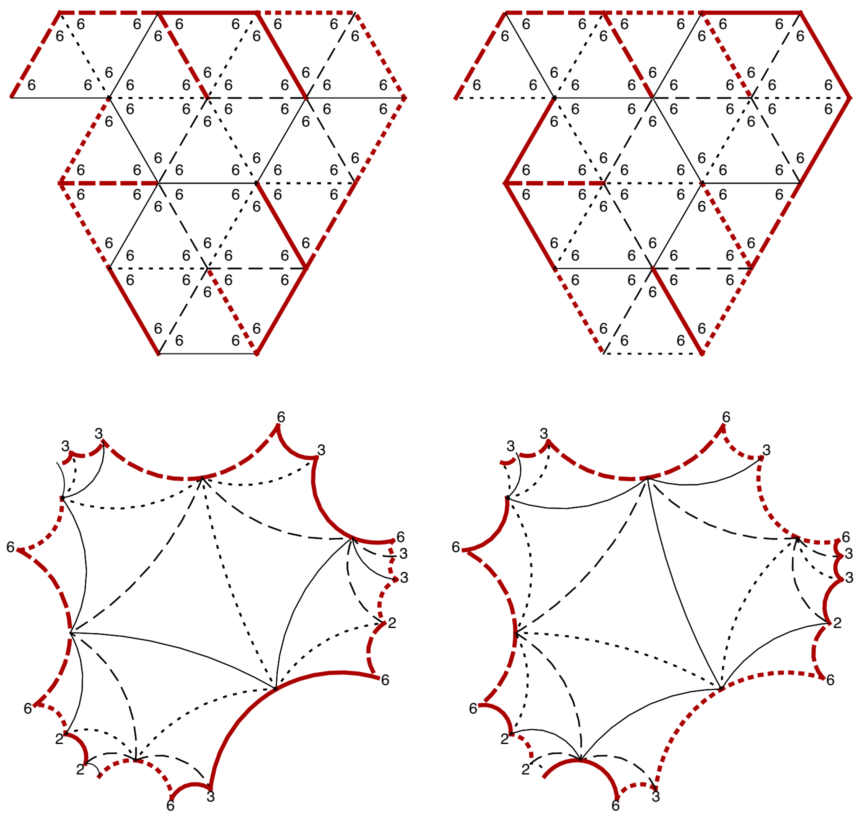
(⋆62424244322, ⋆62443422242)

Figure 27: Pair 15(4)



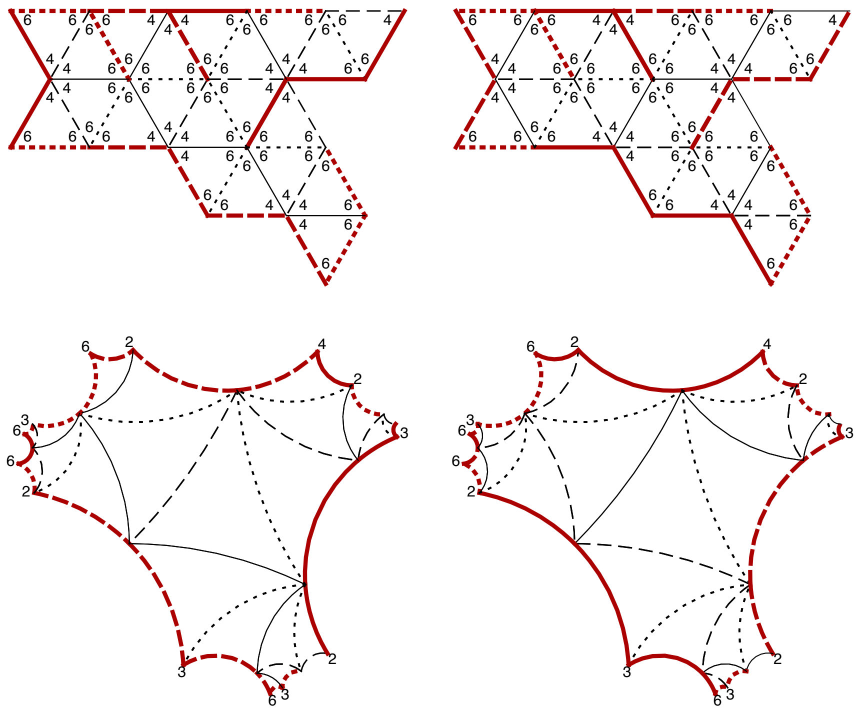
⋆6633626332333

Figure 28: Pair 21(1)



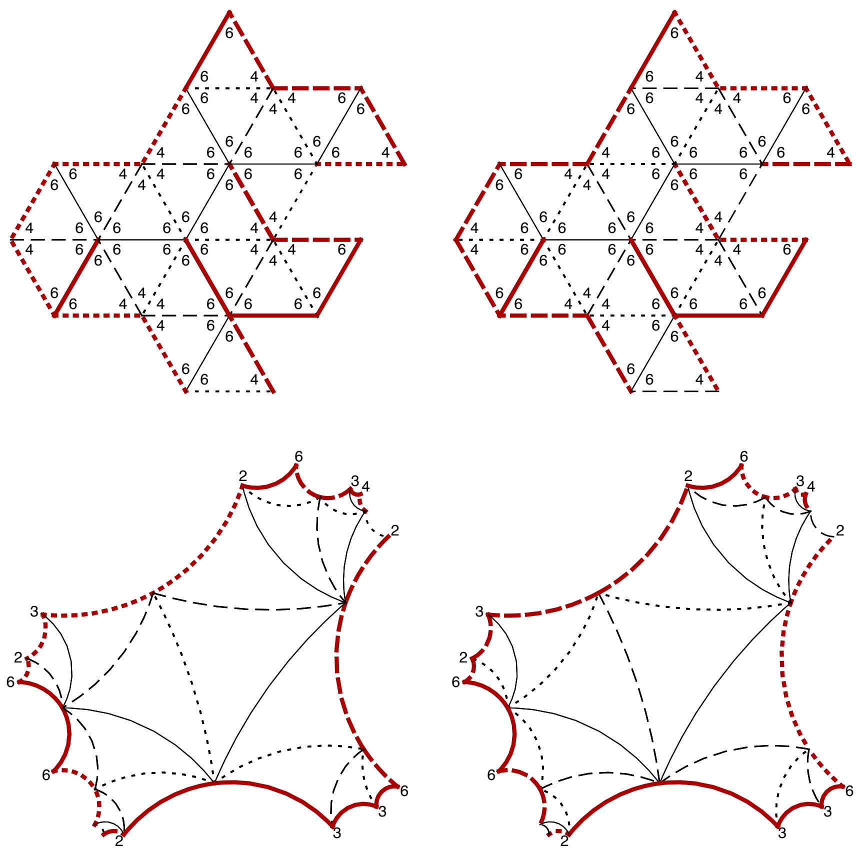
x⋆666362336363322

Figure 29: Pair 21(2)



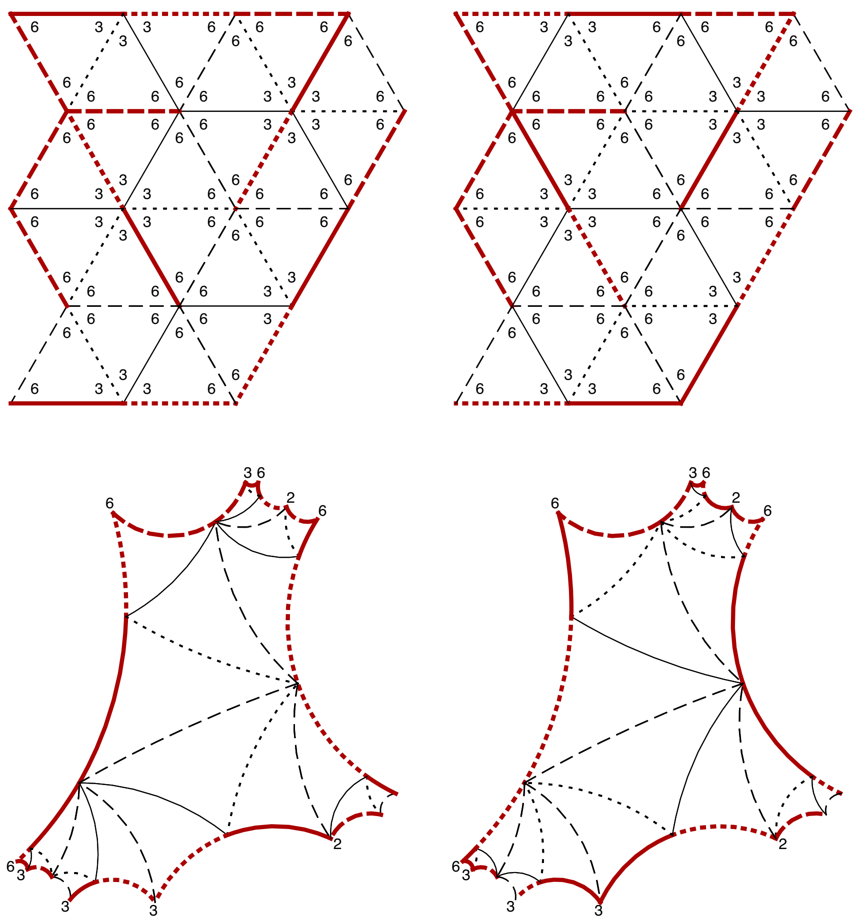
⋆66362423632⋆32

Figure 30: Pair 21(3)



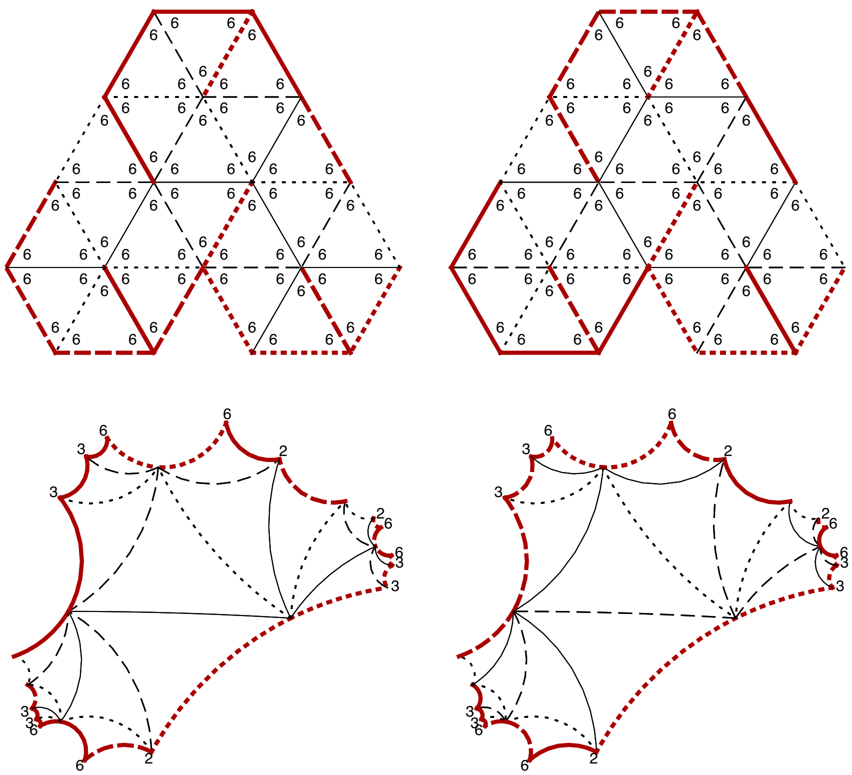
⋆66436232⋆63322

Figure 31: Pair 21(4)



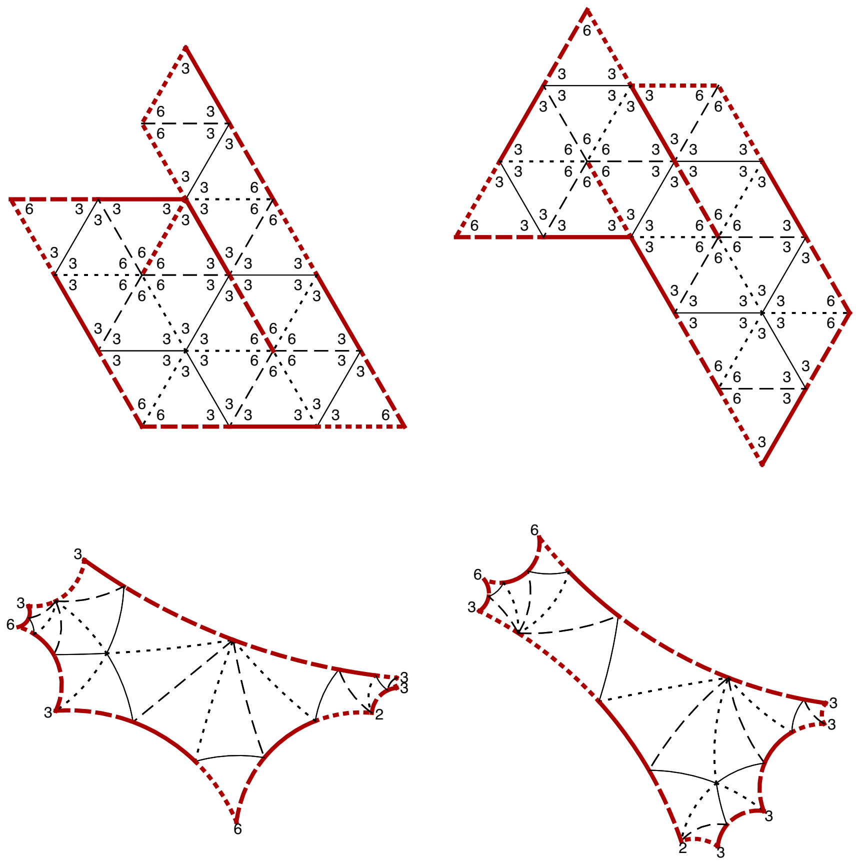
x⋆6636263323

Figure 32: Pair 21(5)



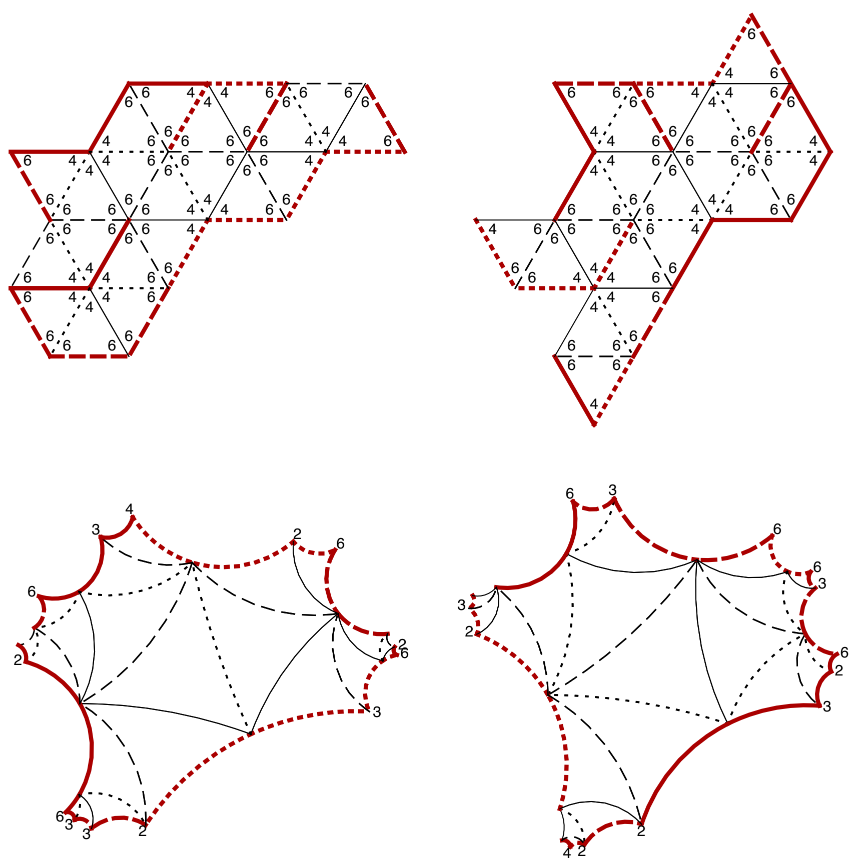
x⋆663326633266332

Figure 33: Pair 21(6)



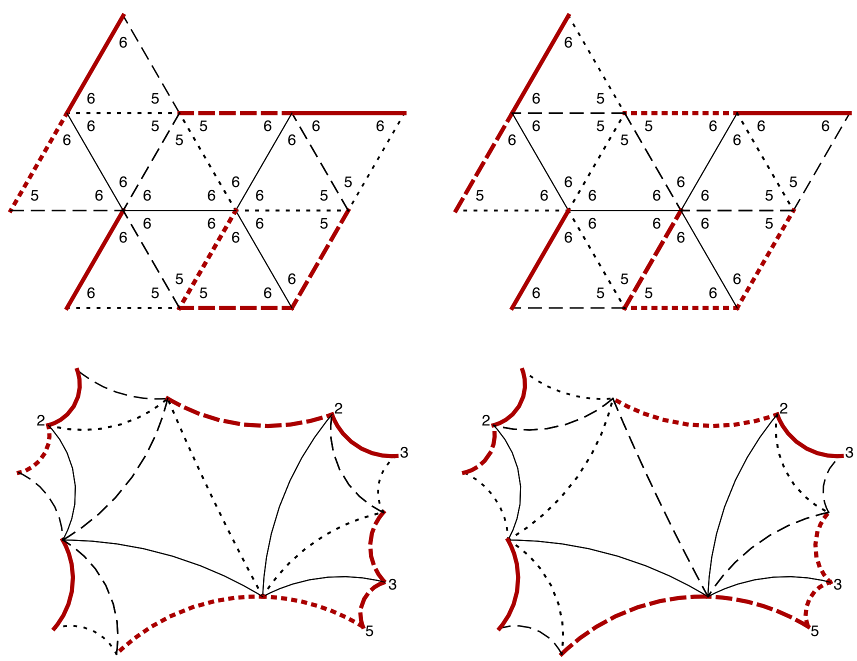
$(\star 63633332,\ \star 66333323)$

Figure 34: Pair 21(7)



(⋆66342⋆63323622, ⋆66364223263⋆32)

Figure 35: Pair 21(8)



x⋆53⋆322

Figure 36: Pair 11f(1)

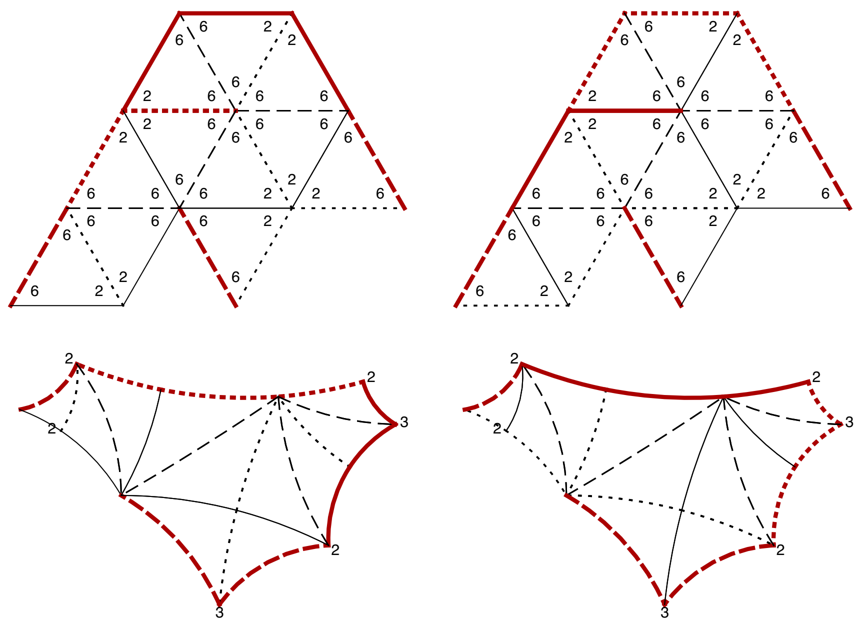


Figure 37: Pair 11f(2)

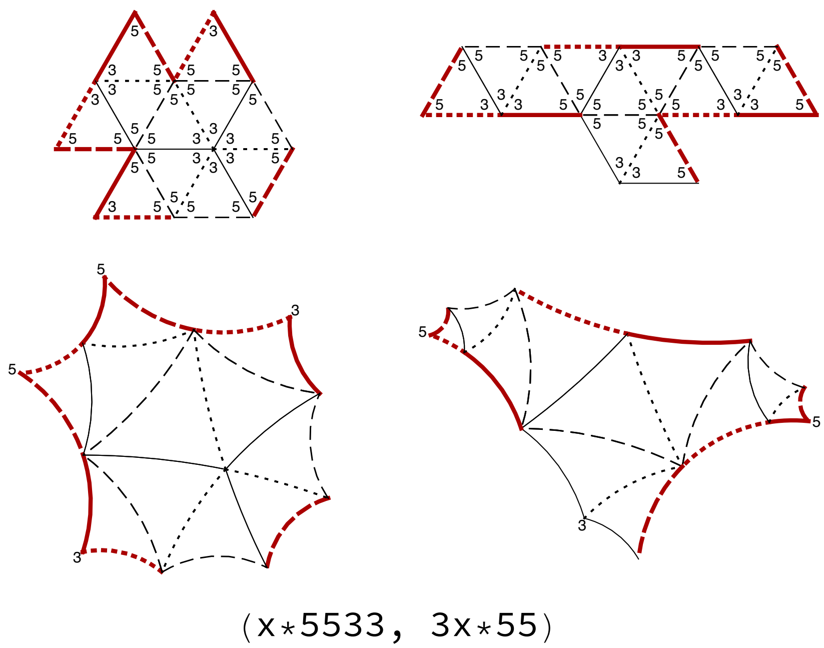
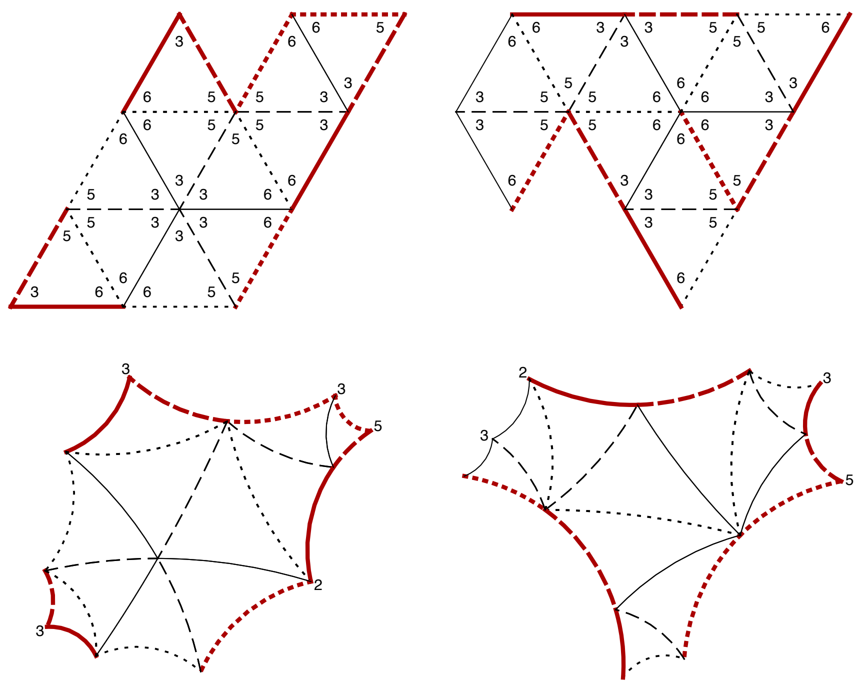
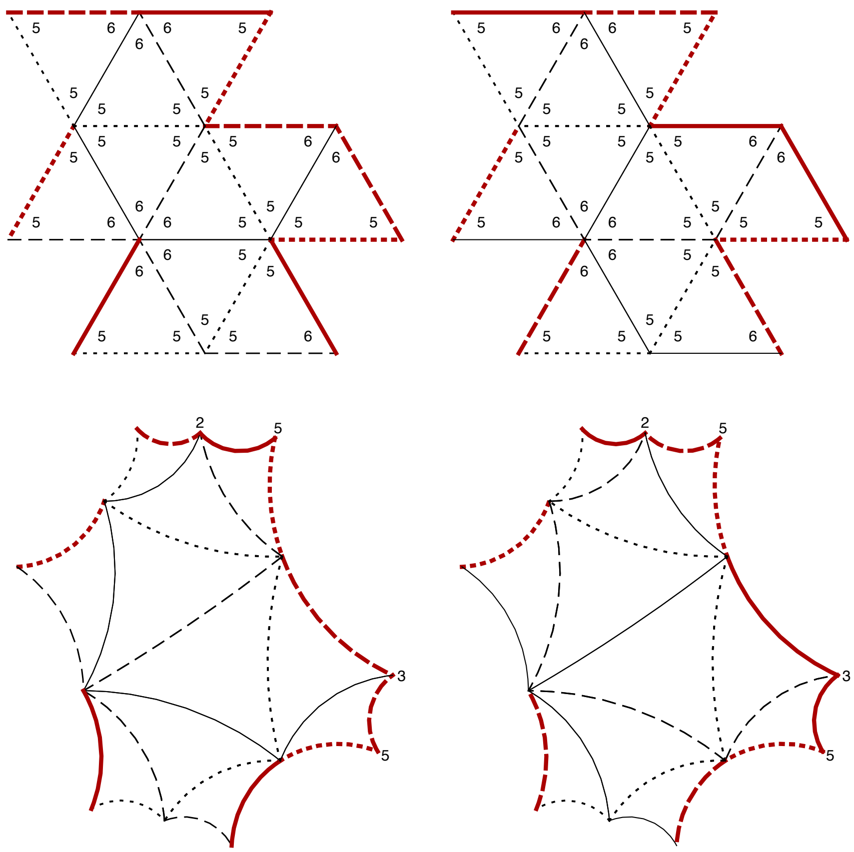


Figure 38: Pair 11f(3)



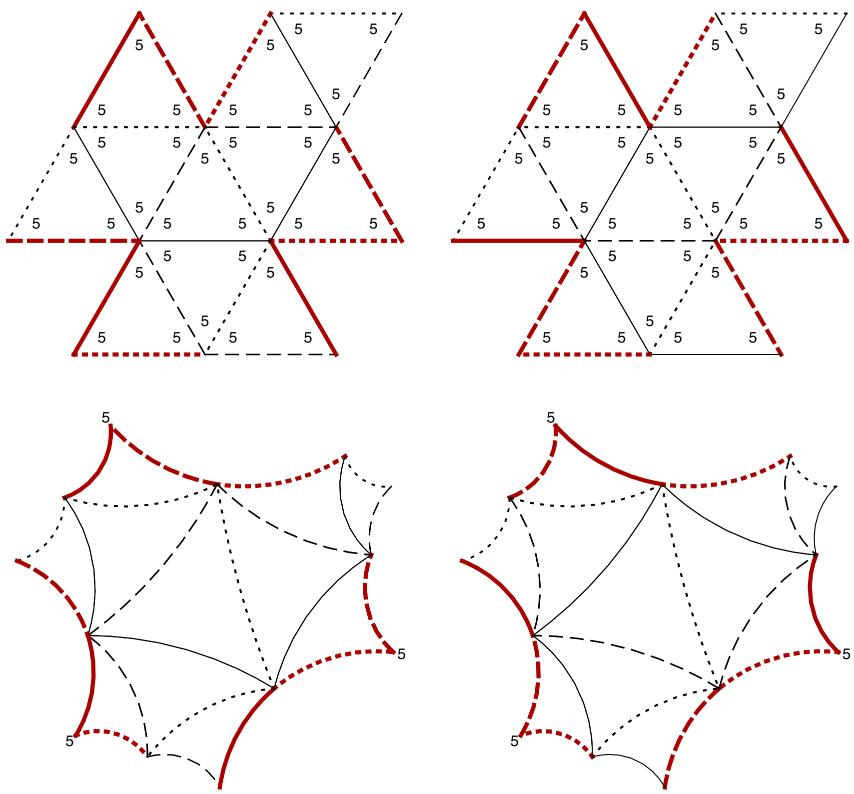
(x⋆53332, 3x⋆532)

Figure 39: Pair 11f(4)



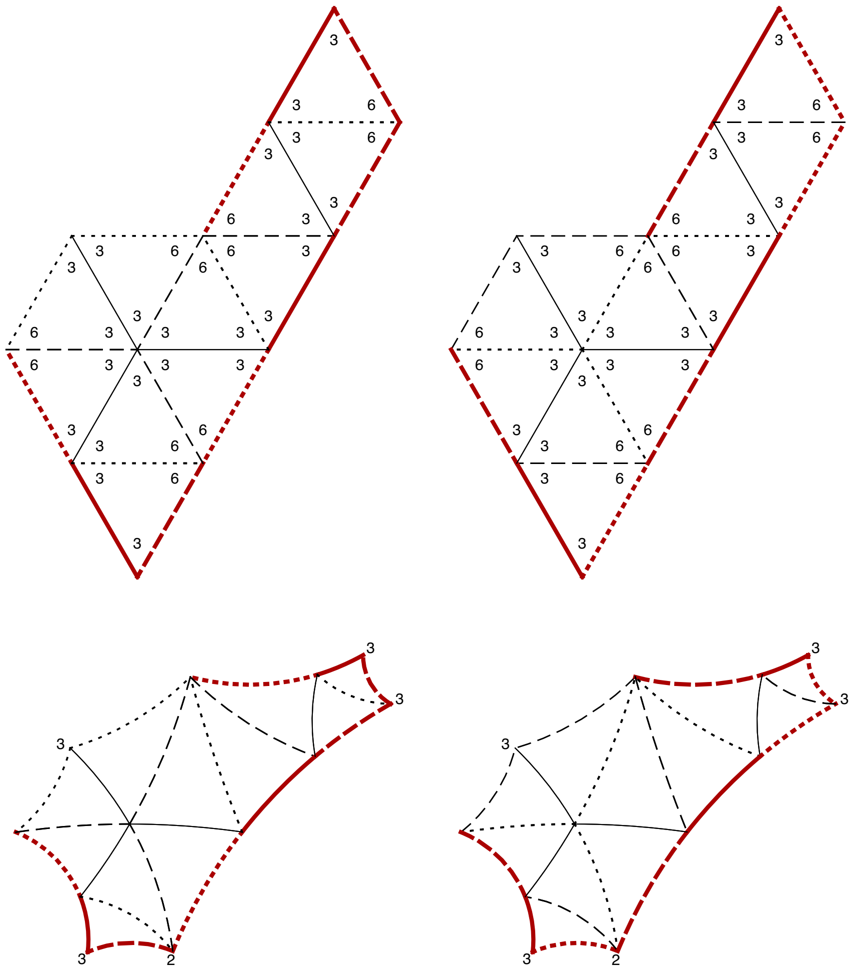
xx⋆5352

Figure 40: Pair 11g(1)



$\star 5 \star 5 \star 5$

Figure 41: Pair 11g(2)



3⋆3332

Figure 42: Pair 11g(3)

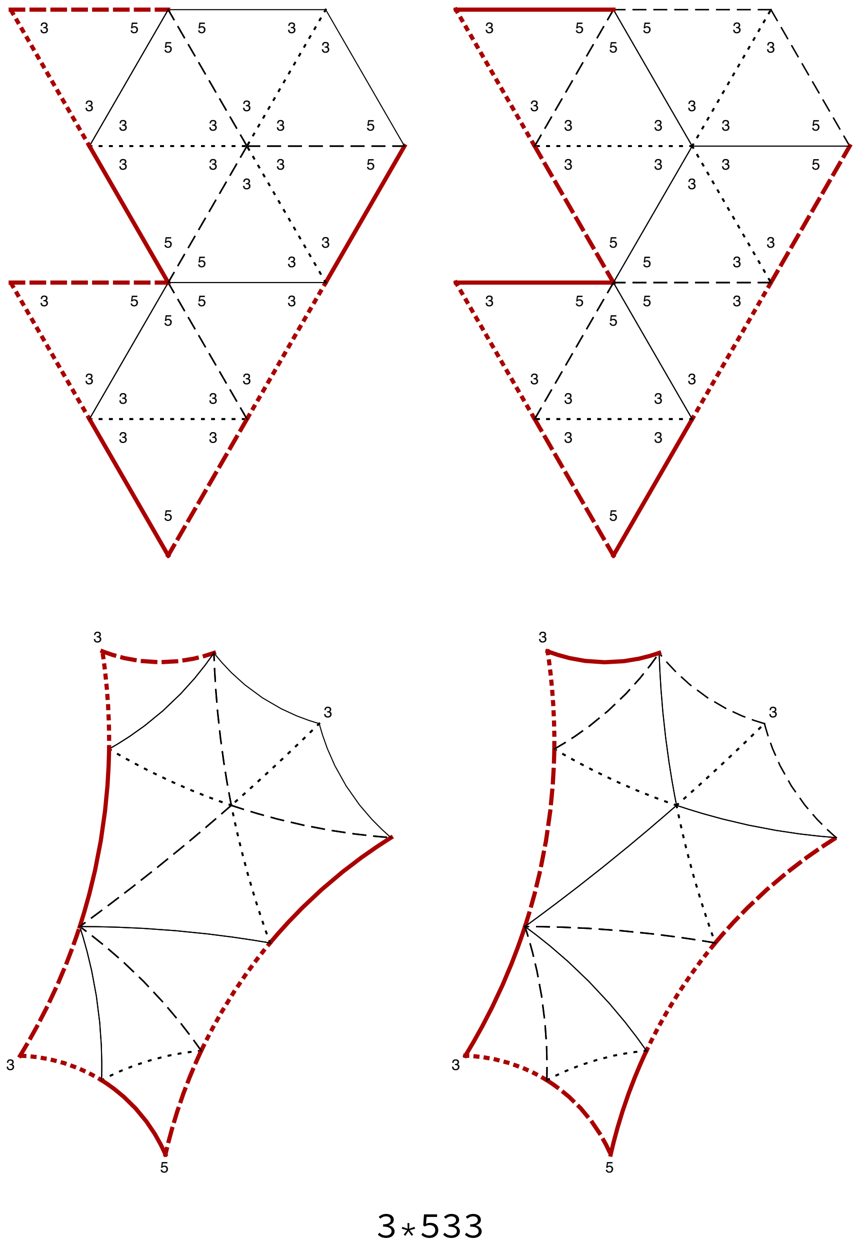
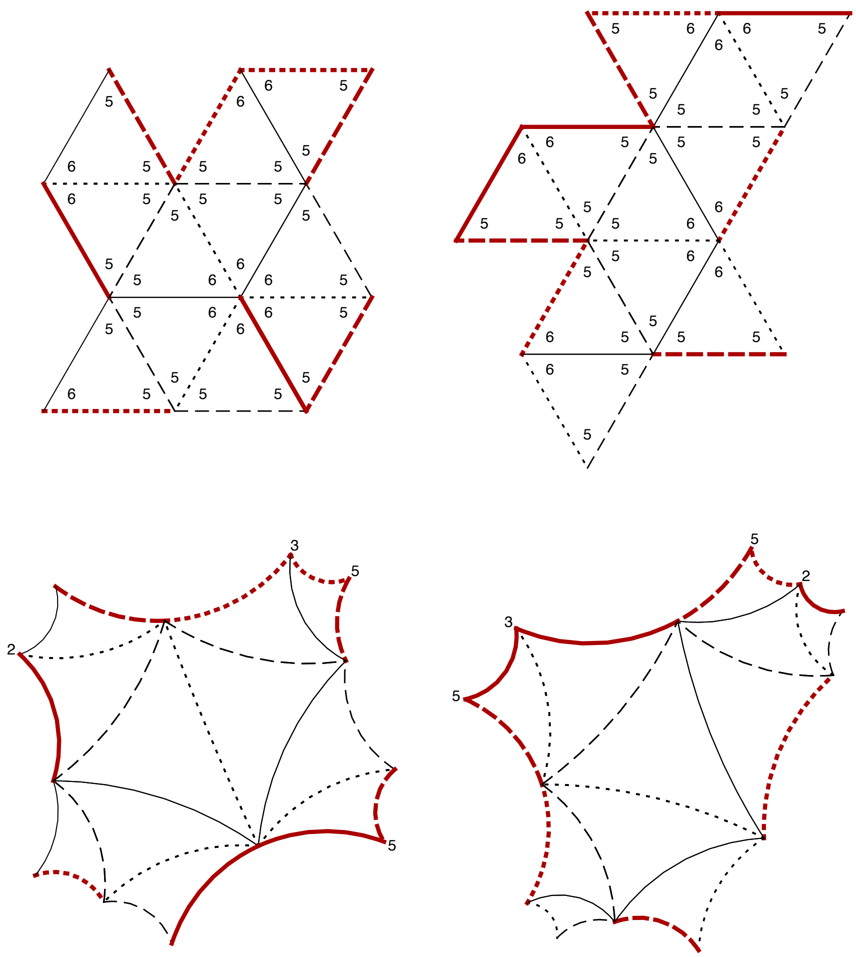
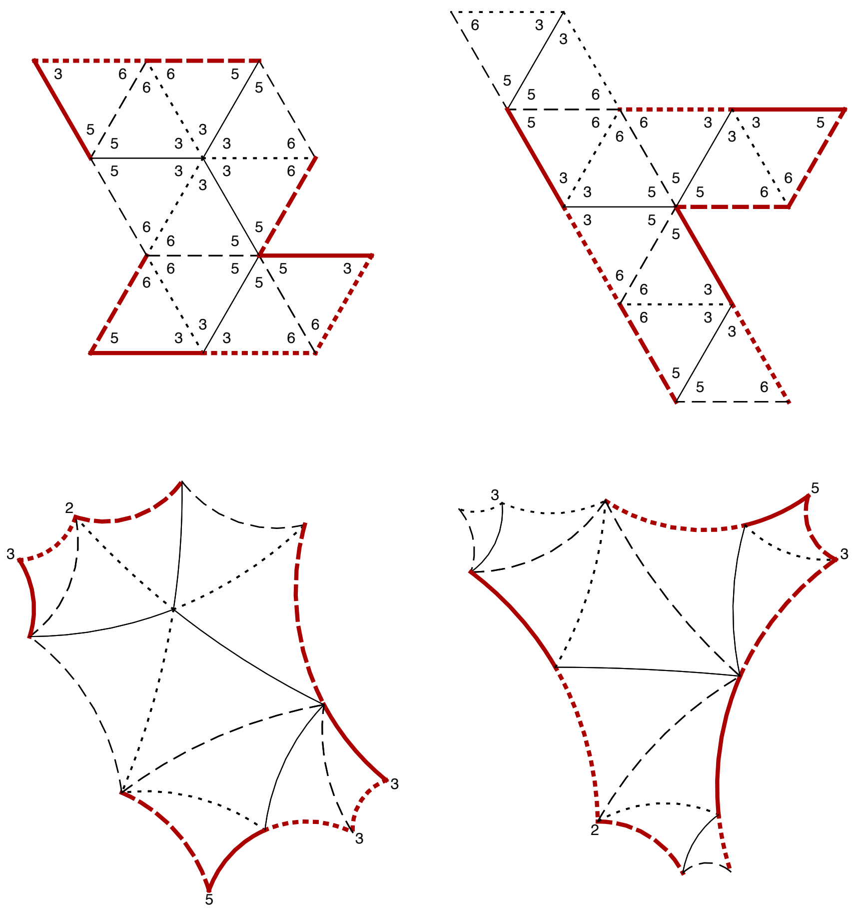


Figure 43: Pair 11g(4)



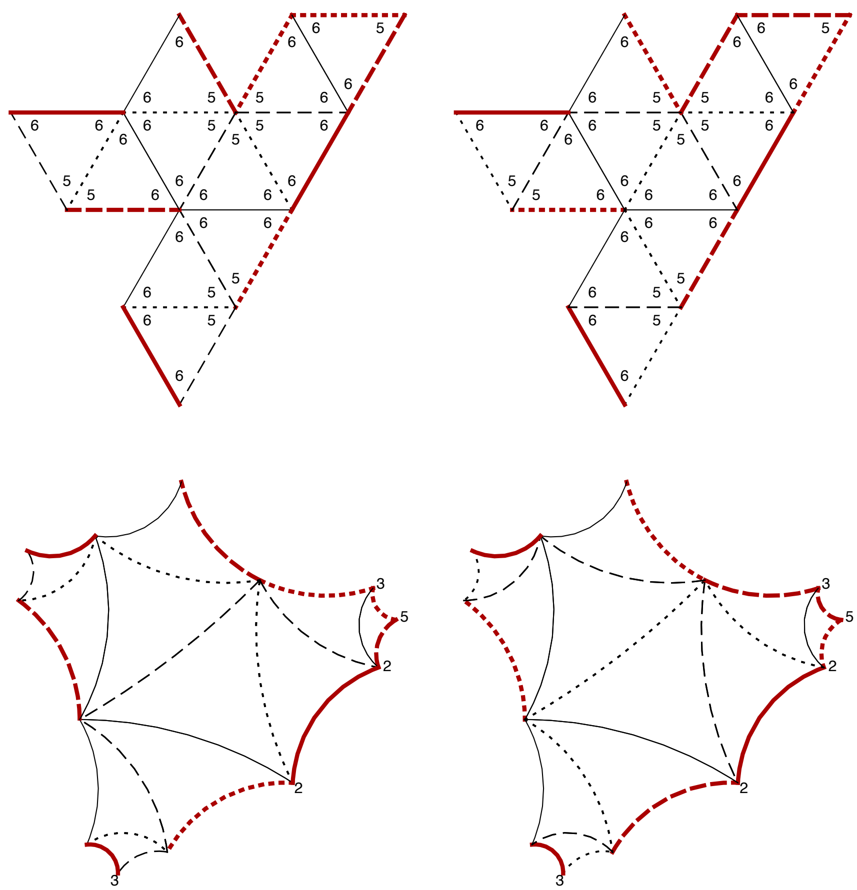
(xx⋆5532, xx⋆5352)

Figure 44: Pair 11g(5)



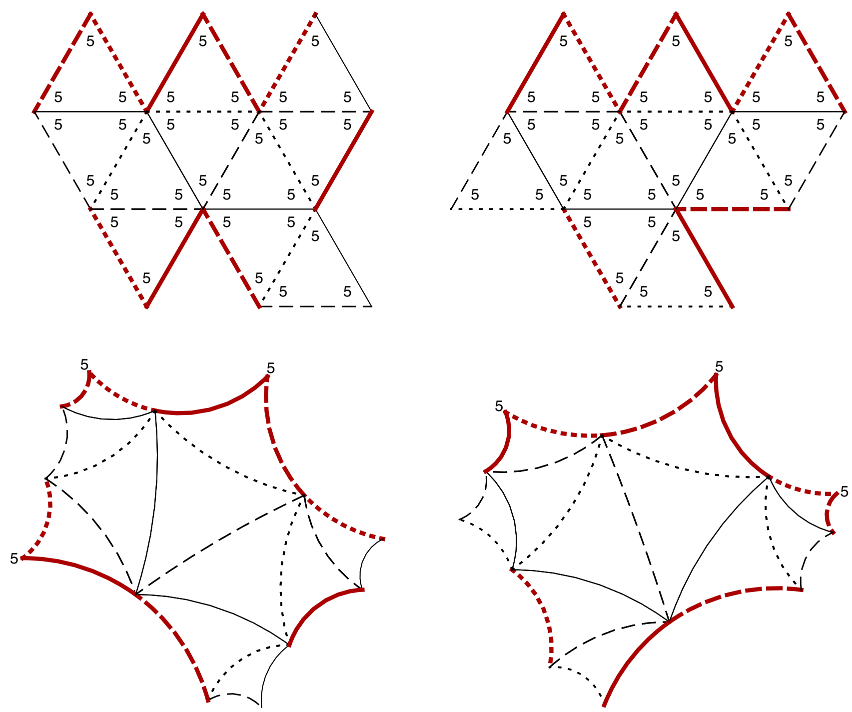
(⋆533⋆32, 3⋆53⋆2)

Figure 45: Pair 11g(6)



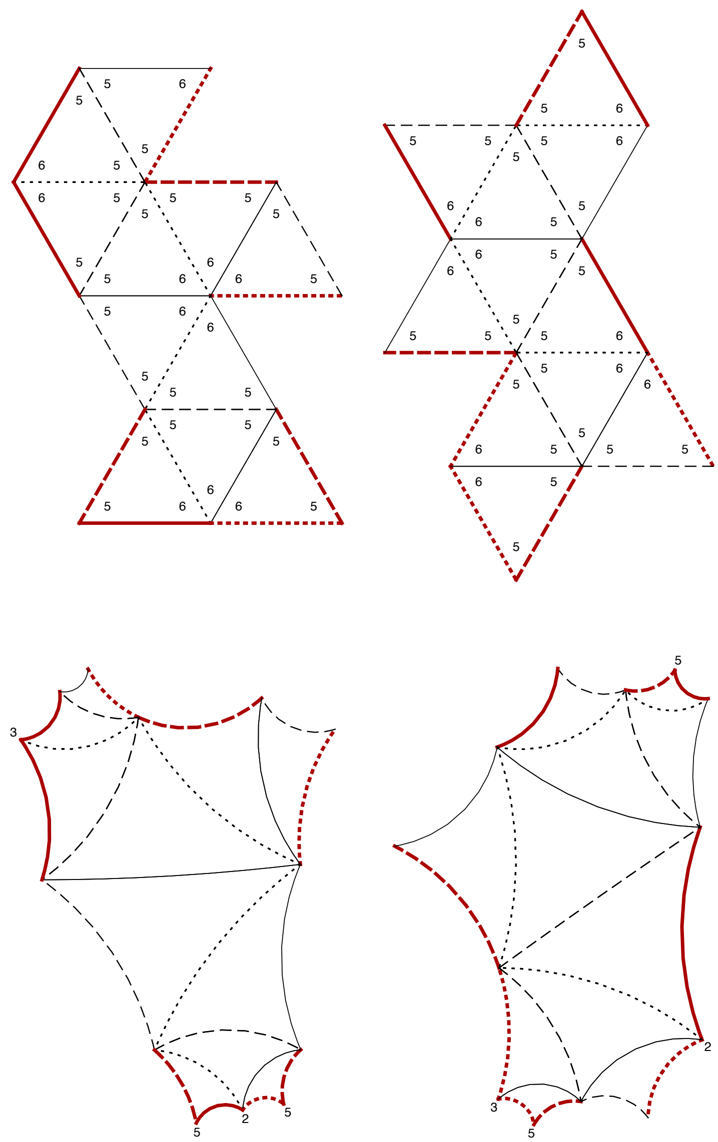
x⋆5322⋆3

Figure 46: Pair 11h(1)



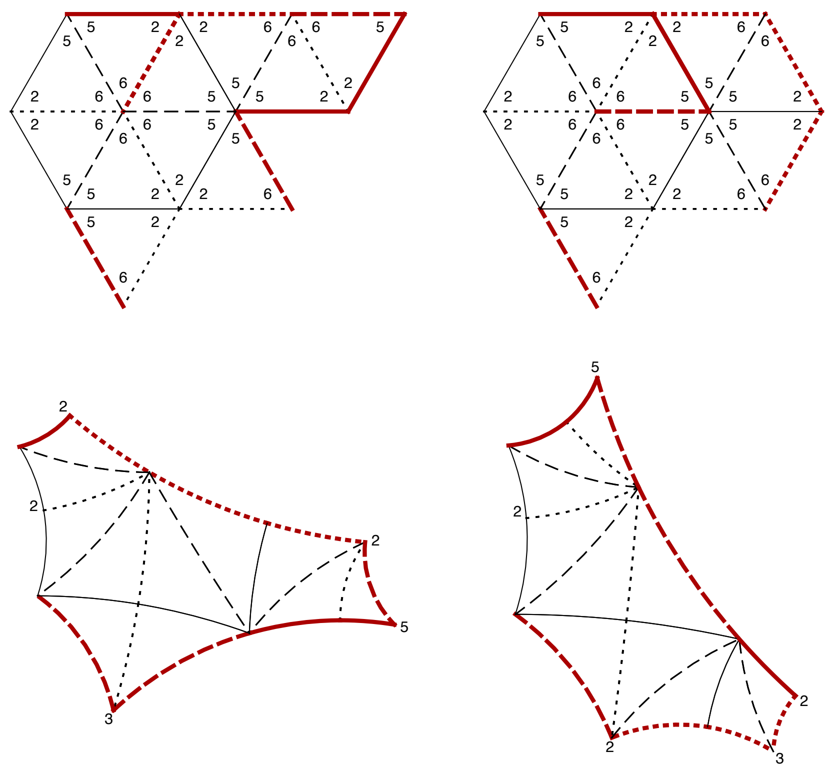
(x⋆55⋆5, x⋆555⋆)

Figure 47: Pair 11h(2)



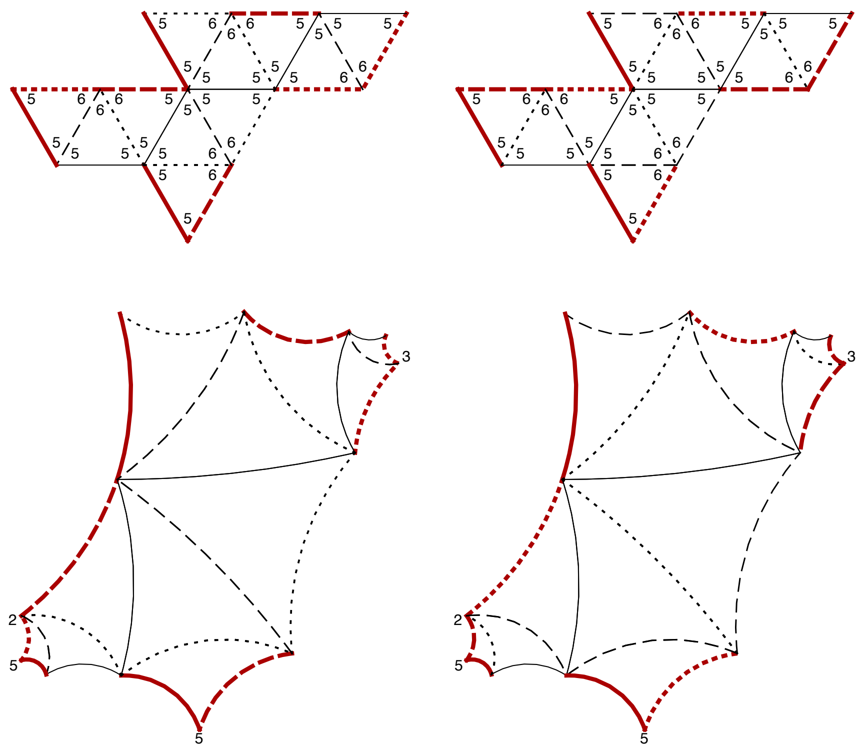
(o⋆5352, o⋆5532)

Figure 48: Pair 11h(3)



(2⋆5322, 2⋆5232)

Figure 49: Pair 11h(4)



xx⋆5532

Figure 50: Pair 11i(1)

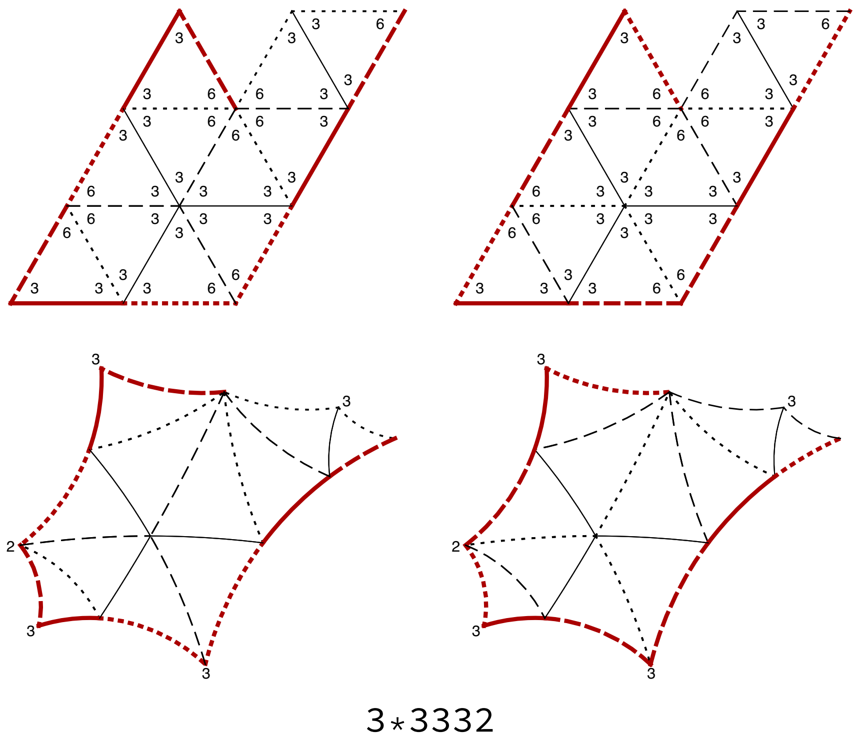
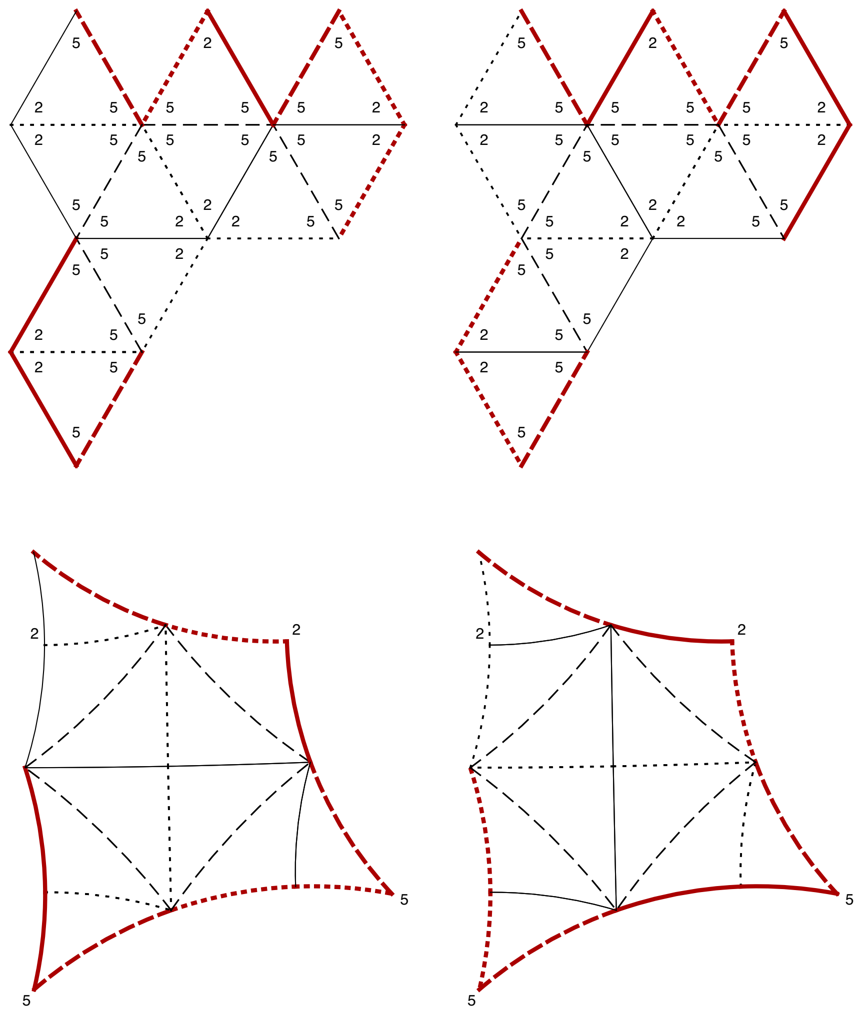


Figure 51: Pair 11i(2)



2⋆552

Figure 52: Pair 11i(3)

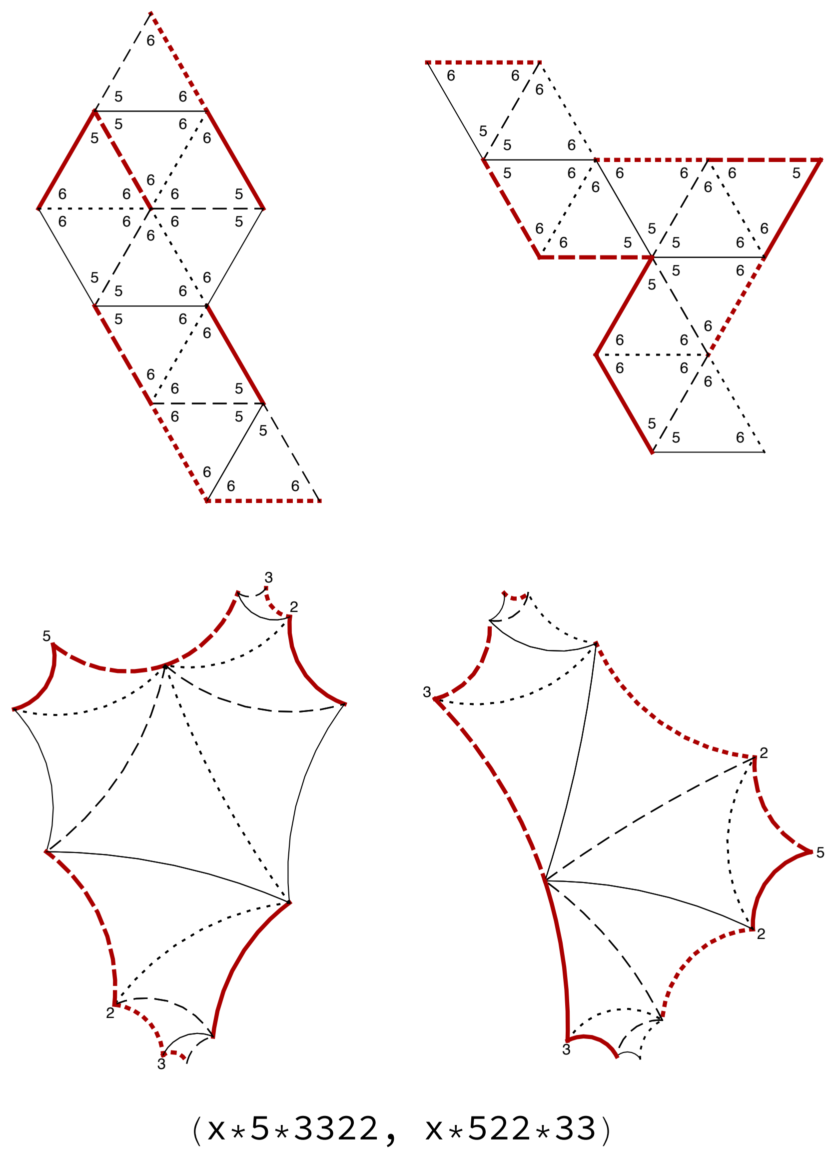
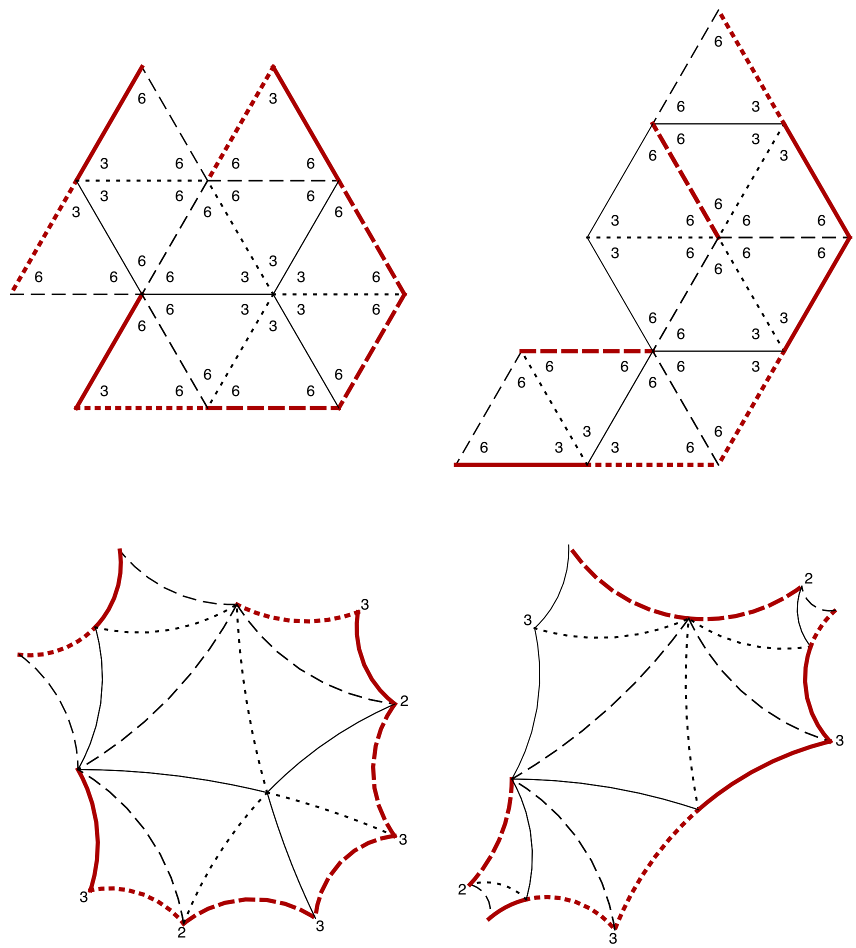


Figure 53: Pair 11i(4)

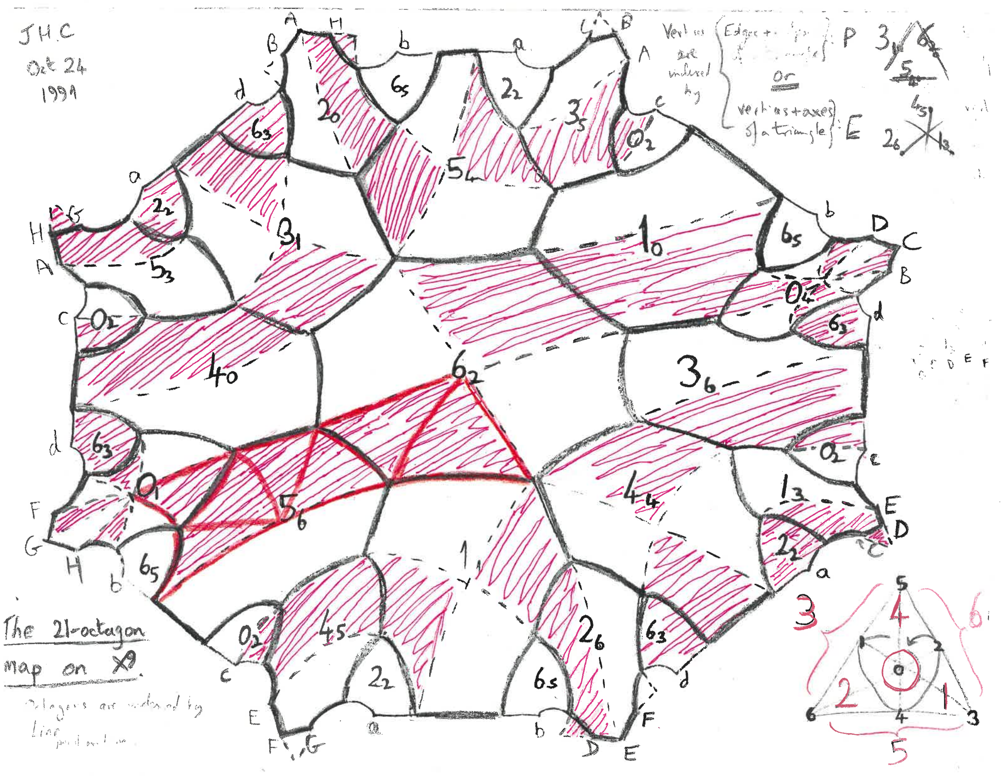


(x⋆332332, 3x⋆3322)

Figure 54: Pair 11i(5)

A The master at work

Here, in the hand of the master, are quilts, along with associated diagrams for isospectral pairs (including peacocks rampant and couchant in the hand of the pupil).



The 21-octagon map on $\times^9$. Octagons are indexed by $\text{line}_{\text{point on line}}$.

No.	generators	K_1	G_1	A_1,B_1	notes
7_1	a,b,c	x^23	*444	*424242	
7_2	a,b′,c	x^16	*443	*42423	a′ = cac
7_3	a′,b′,c	x^9	*433	*4233	b′ = aba
13_1	d,e,f	x^704	*444	*422422422	
13_2	d,e′,f	x938	*644	*6622342242	e′ = ded
13_3	d′,e′,f	x1172	*664	*62234263662	d′ = fdf
13_4	d′,e′,f′	x938	*663	*633626362	f′ = e′fe′
13_5	d′,e″,f′	x470	*633	*663332	e″ = d′e′d′
13_6	g,h,i	x^1406	*666	*632663266326	
13_7	g,h′,i	x938	*663	*632666233 ← NB	h′ = ghg
13_8	g′,h′,i	x704	*643	*63436222,*62633224	g′ = igi
13_9	g′,h′,i′	x938	*644	*6262242243	i′ = g′ig′
15_1	j,k,l	x3362	*663	*63362333222	
15_2	j,k,l′	x4202	*664	*6262234342242	l′ = jlj
15_3	j′,k,l′	x3362	*644	*62234434242,*63422243442	j′ = kjk
15_4	j′,k′,l′	x2522	*444	*444222442	k′ = l′kl′
21_1	p,q,r	x^1682	*633	NB. *63633332, *66333323 NB	

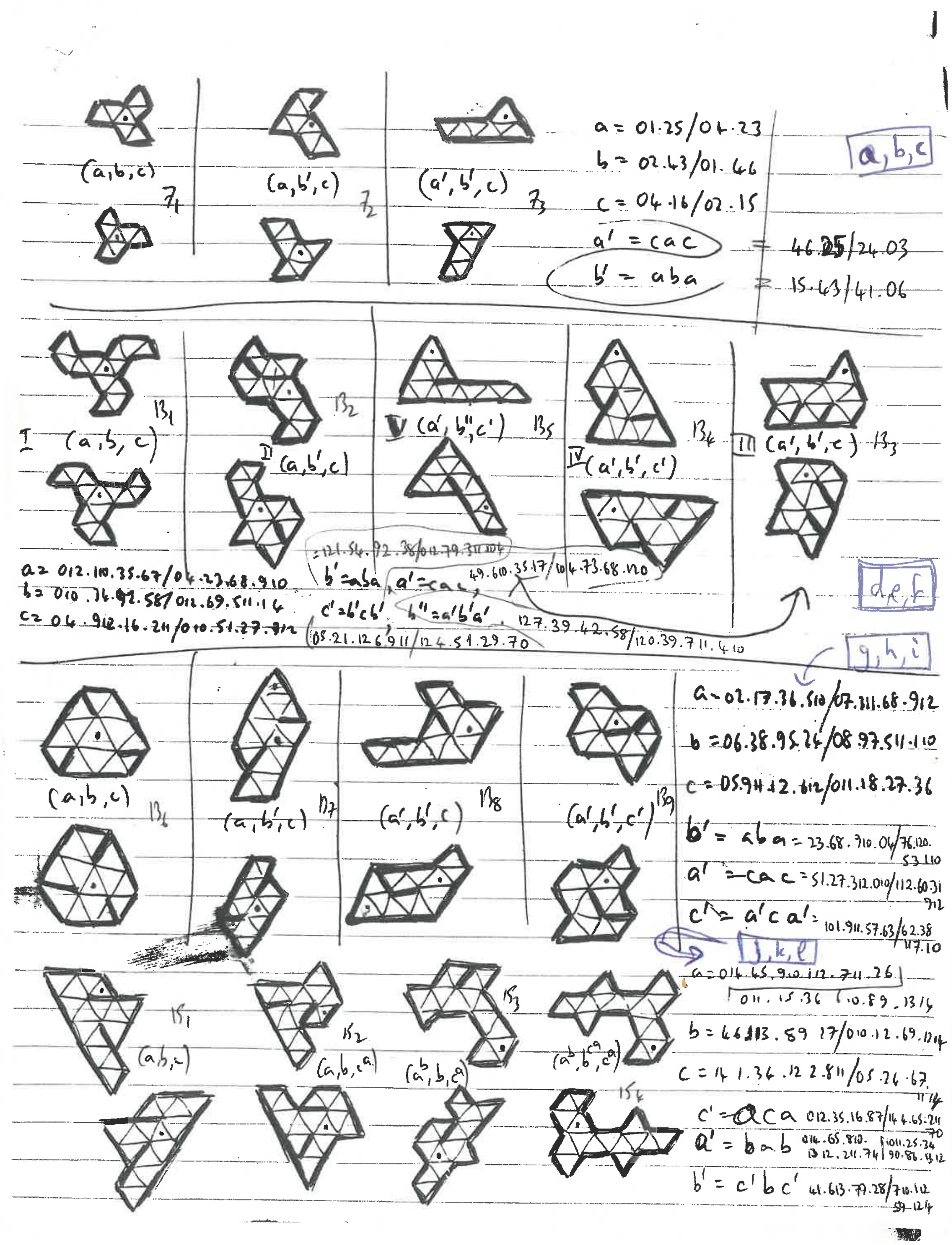
① Add perms a, b, c, d, e, f, g, h, i, j, k, l underneath

② Add subscripts to headings

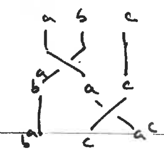
③ Correct 3 errors!

④ Add last line to table!

⑤ Enter rest of K_1 column!



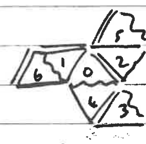
(a,b,c)
(a,b',c)
(a',b',c)
a = 01.25/04.23
b = 02.43/01.46
c = 04.16/02.15
a' = cac
b' = aba
a,b,c
d,e,f
g,h,i
j,k,l



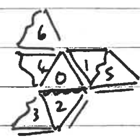
01.25 / 04.23 = a a^c = 46.25 / 24.03

02.43 / 01.46 = b b^a = 15.43 / 41.06

04.16 / 02.15 = c



a, b, c

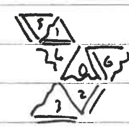


*444

a, b^a, c

(a, b', c)

b' = aba

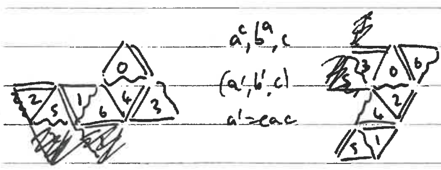


*443

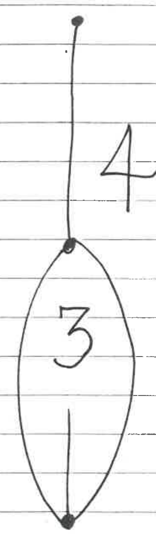
a^c, b^a, c

(a', b', c)

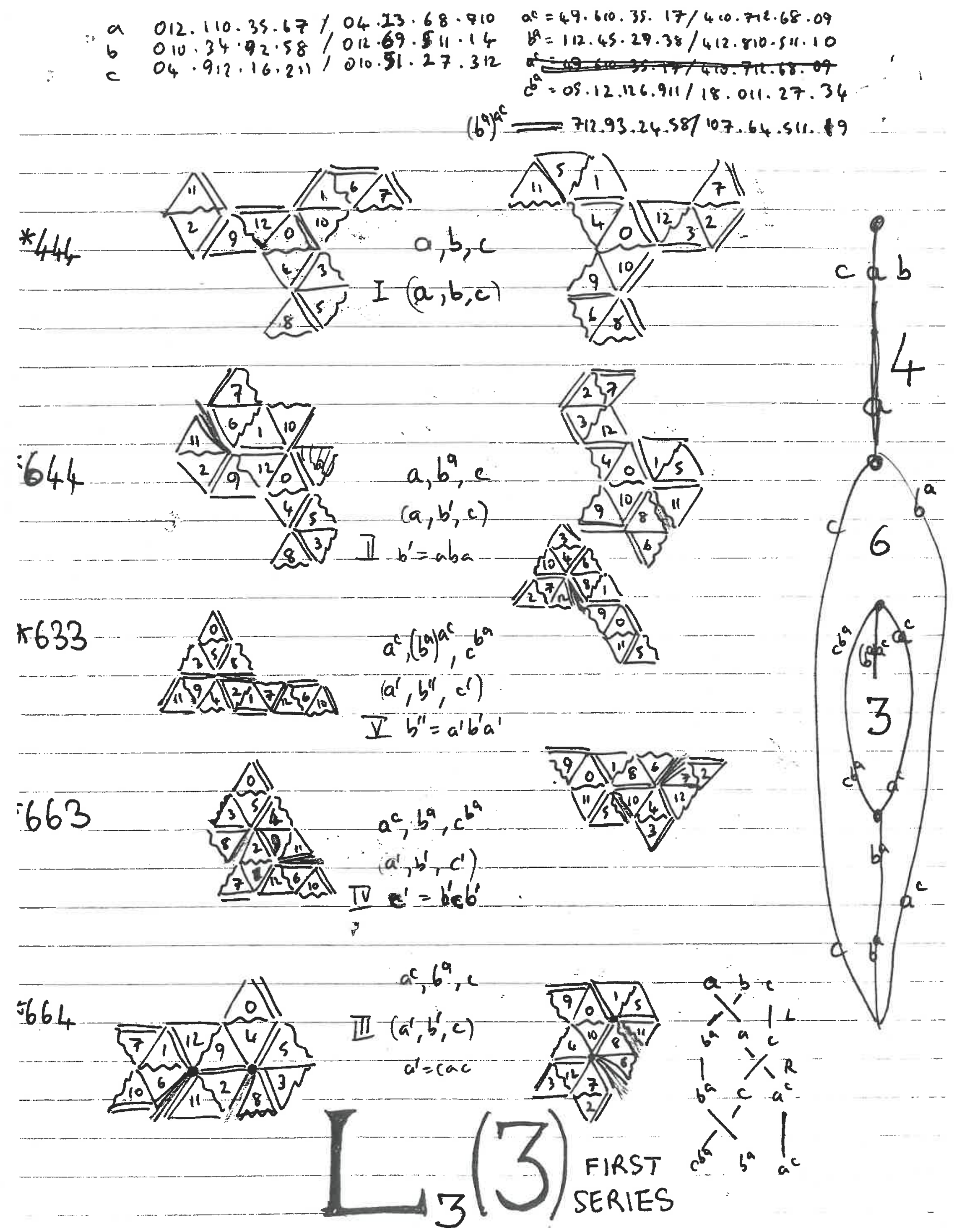
a' = cac



*433



$L_3(2)$


*444
a, b, c
I (a, b, c)
644
a, b^a, c
(a, b', c)
II b' = aba
*633
V b'' = a'b'a'
663
a^c, b^a, c^{b^a}
(a', b', c')
IV c' = b'cb'
664
a^c, b^a, c
III (a', b', c)
a' = cac
c a b
4
6
3
L3(3) FIRST SERIES

02.17.36.510/07.311.68,912 = a

06.38.95.24/08.97.511.110 = b

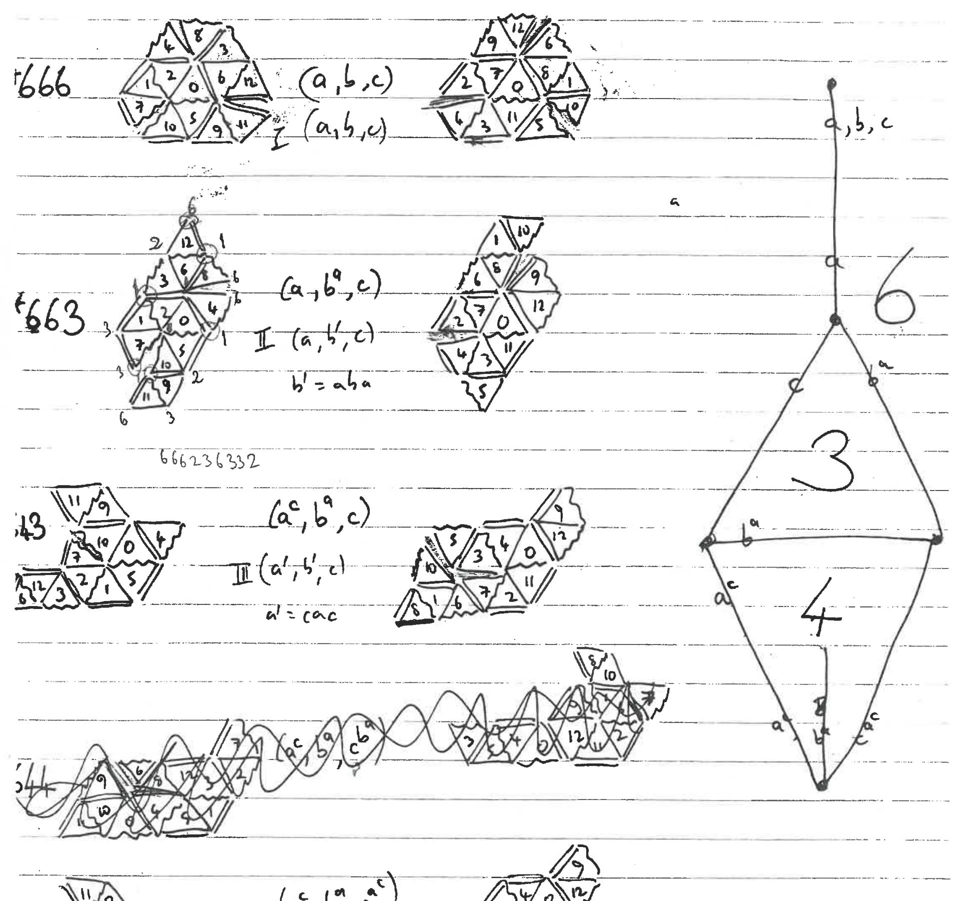
05.911.12.611/011.18.27.34 = c

a^c = 51.27.312.010/211.04.16.912

b^a = 23.68.109.04/67.120.53.110

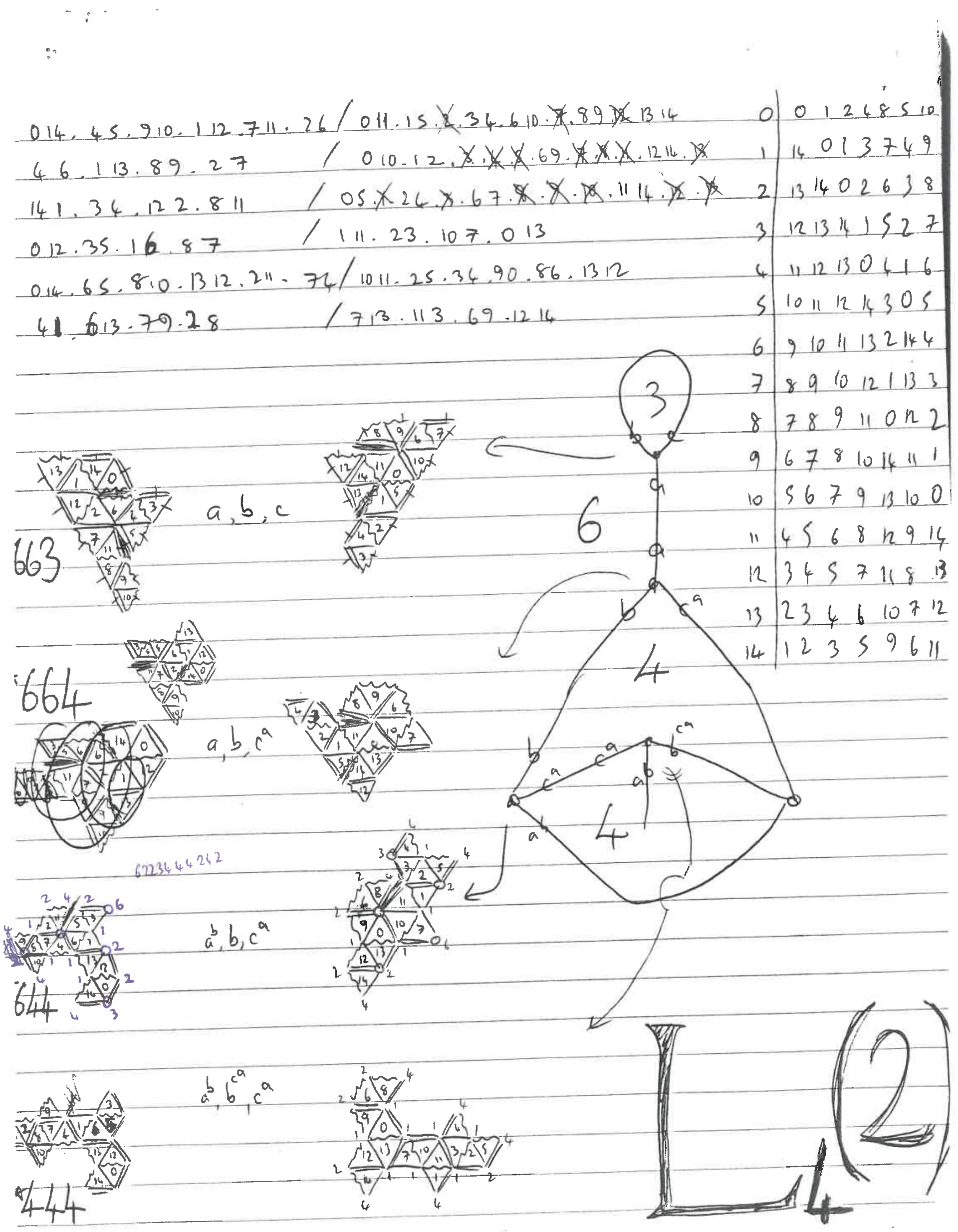
~~c^a = 45.1011.13.812/112.810.26.54~~

c^{a^c} = 110.911.57.63/24.68.117.30



$L_3(3)$

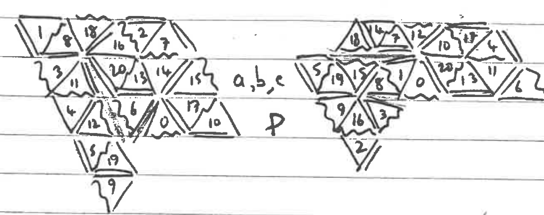
SECOND SERIES

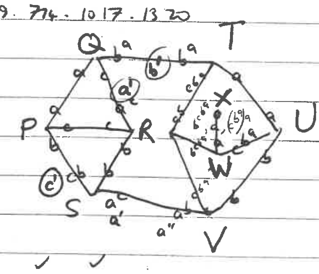

663
664
644
444
a, b, c
L4(2)

a = 15 17 . 5 12 . 13 14 . 8 18 . 3 11 . 7 2 . 16 20 / 0 1 . 4 17 . 7 12 . 9 16 . 10 20 . 11 13 . 15 19

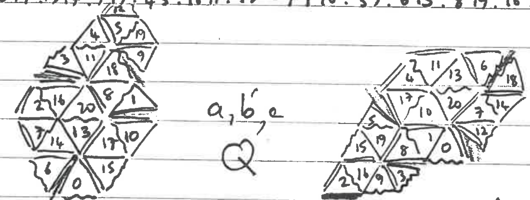
b = 16 18 . 6 13 . 14 15 9 19 . 4 12 . 8 3 . 17 0 / 0 20 . 3 16 . 6 11 . 8 15 . 9 19 . 10 12 . 14 18

c = 1 8 . 2 16 . 4 11 . 5 19 . 7 14 . 10 17 . 13 20 / 1 8 . 2 16 . 6 11 . 5 19 . 7 14 . 10 17 . 13 20

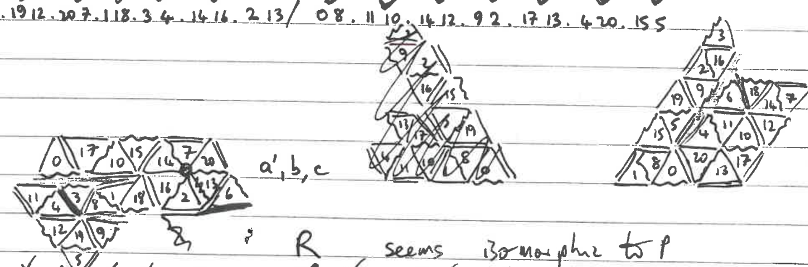




b′ = b^a = 20 8 . 6 14 . 13 17 . 9 19 . 4 5 . 18 11 . 15 0 / 1 10 . 3 9 . 6 13 . 8 19 . 16 15 . 20 7 . 14 18

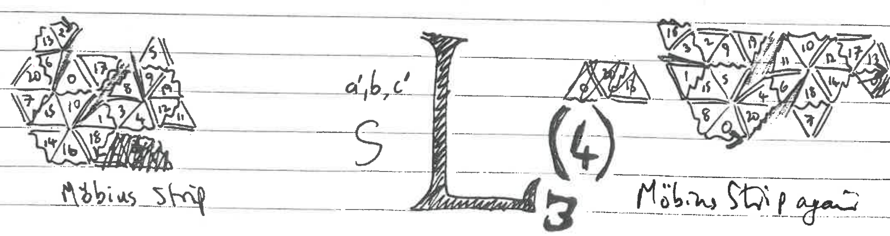


a′ = a^c = 15 10 . 19 12 . 20 7 . 1 18 . 3 4 . 14 16 . 2 13 / 0 8 . 11 10 . 14 12 . 9 2 . 17 13 . 4 20 . 15 5



R seems isomorphic to P

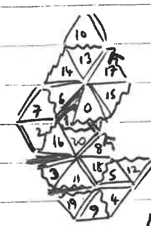
c′ = c^b = 1 3 . 2 18 . 12 11 . 5 9 . 7 15 . 10 0 . 6 20 / 1 15 . 2 3 . 4 6 . 5 9 . 7 18 . 12 17 . 13 0



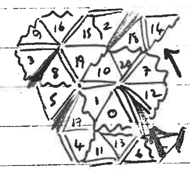
$c'' = c^{b^a} = 1\,20 \,.\, 2\,16 \,.\, 5\,18 \,.\, 4\,9 \,.\, 7\,6 \,.\, 10\,13 \,.\, 17\,8 \,/\, 10\,19 \,.\, 2\,15 \,.\, 4\,11 \,.\, 5\,8 \,.\, 20\,18 \,.\, 1\,17 \,.\, 6\,7$

a, b', c''

T



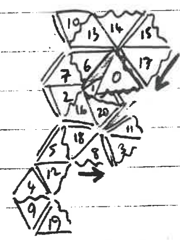
Möbius Strip with overlap



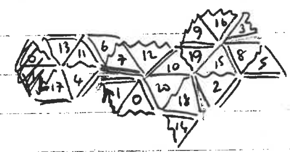
Möbius Strip again

a, b, c''

U



Annulus.

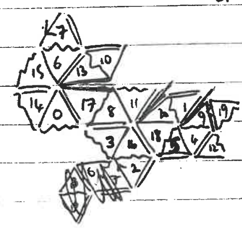


Annulus.

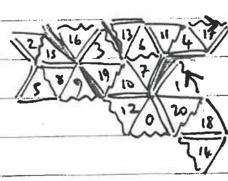
$a'' = a^b = 14\,0 \,.\, 5\,4 \,.\, 6\,15 \,.\, 3\,16 \,.\, 8\,11 \,.\, 7\,2 \,.\, 18\,20 \,/\, 20\,1 \,.\, 4\,17 \,.\, 7\,10 \,.\, 19\,3 \,.\, 12\,0 \,.\, 6\,13 \,.\, 8\,9$

a'', b, c''

V

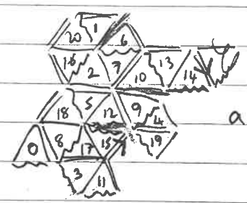


Annulus.



Annulus

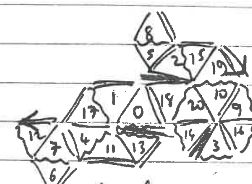
$b'' = b^{c^{b^a}} = 2\,5 . 7\,10 . 14\,15 . 4\,19 . 9\,12 . 17\,3 . 8\,0 / 0\,18 . 3\,16 . 7\,4 . 5\,2 . 9\,10 . 19\,12 . 14\,20$



Annulus

a, b'', c''

W

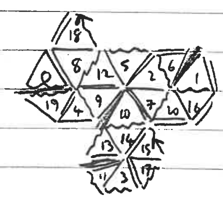


Annulus

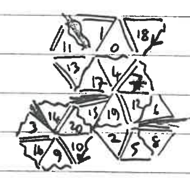
$c''' = (c^{b^a})^a = 1\,16 . 7\,20 . 12\,8 . 4\,9 . 2\,6 . 10\,14 . 15\,18 / 20\,15 . 2\,19 . 17\,13 . 5\,8 . 10\,18 . 0\,4 . 6\,12$

$a\ b''\ c'''$

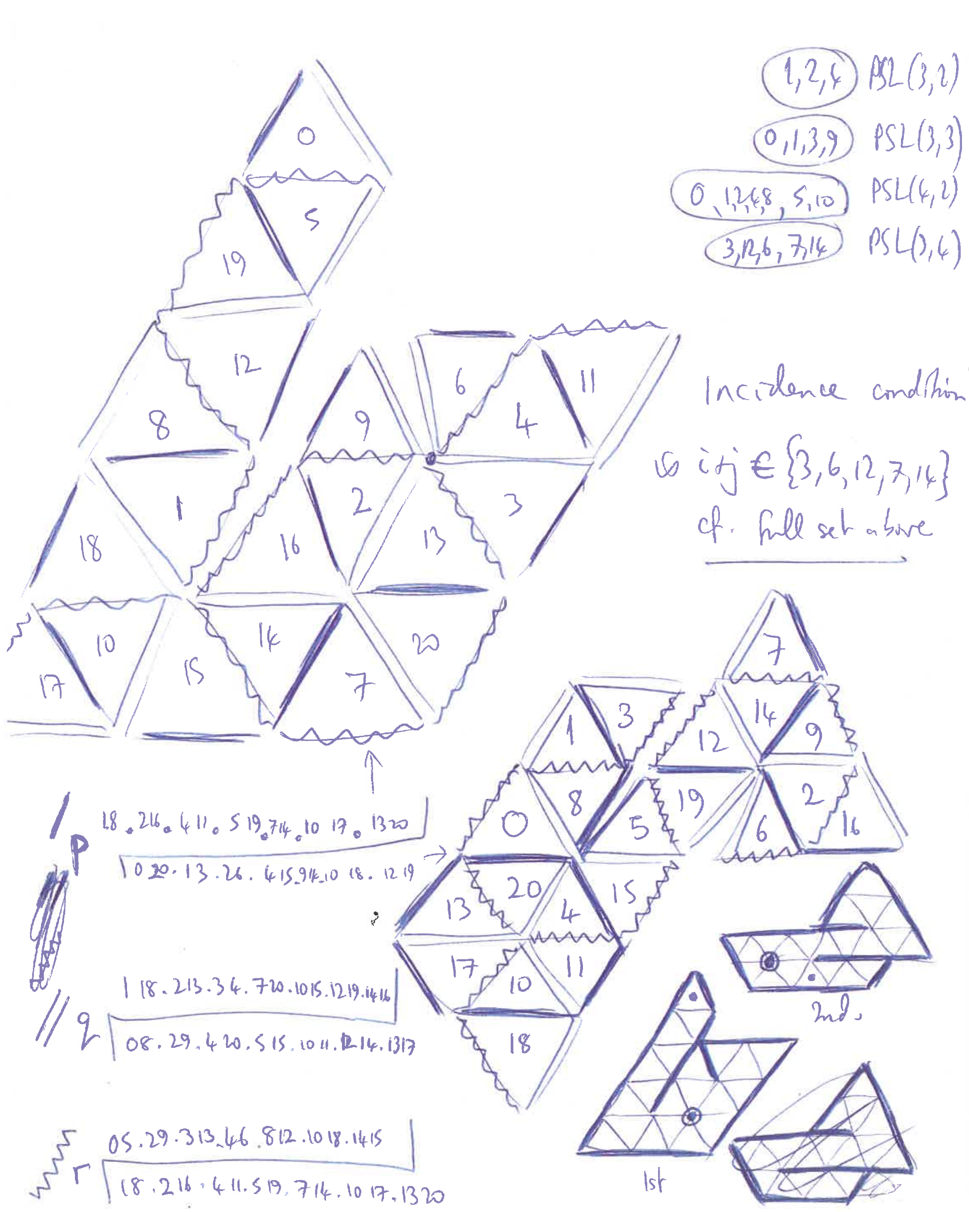
X



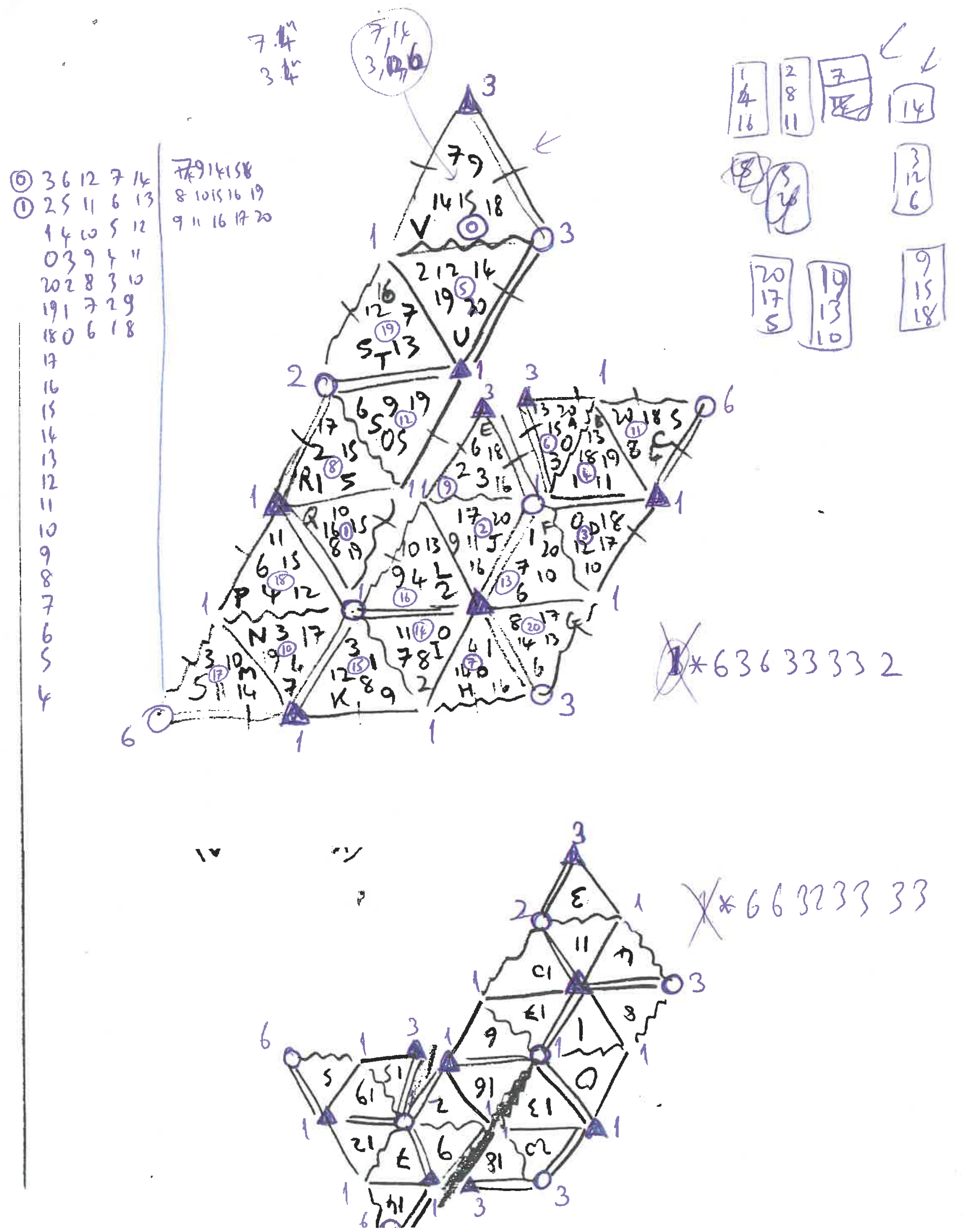
Möbius Strip.



(A very sweet one!)

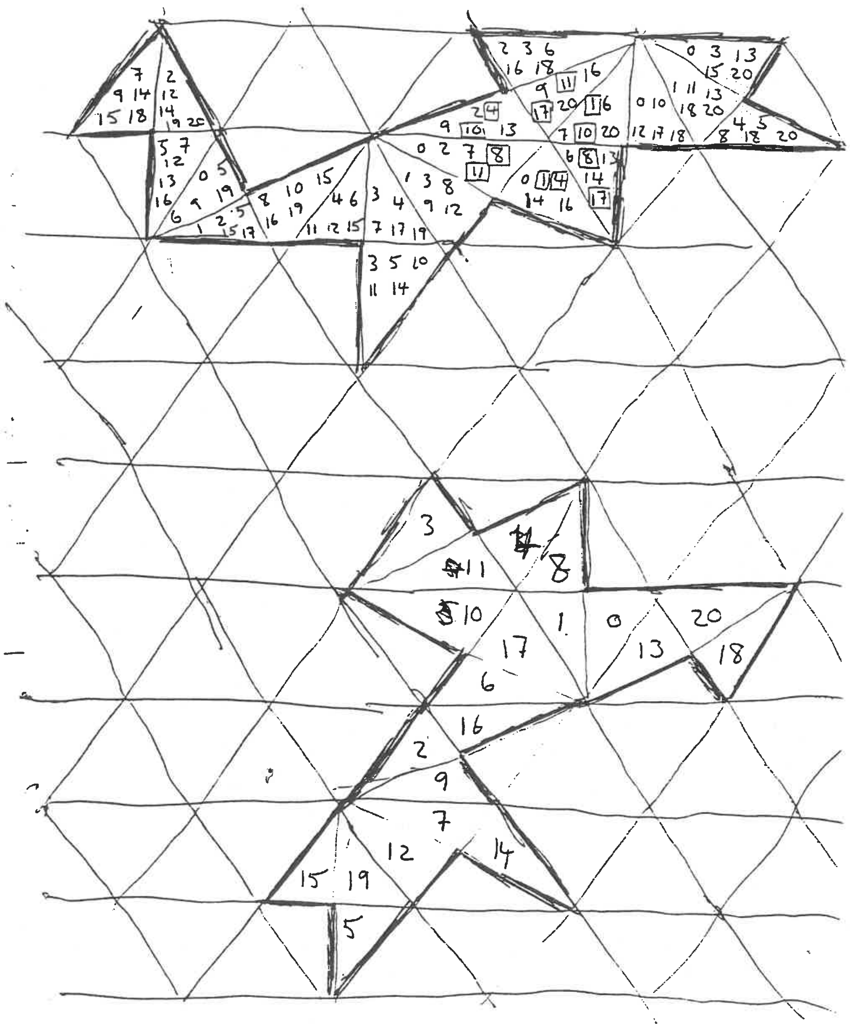


(1,2,4) PSL(3,2)
(0,1,3,9) PSL(3,3)
(0,1,2,4,8,5,10) PSL(4,2)
(3,12,6,7,14) PSL(3,4)
Incidence condition
i+j ∈ {3,6,12,7,14}
cf. full set above
p 18.216.411.519.714.1017.1320
1020.13.26.415.914.1018.1219
q 118.213.34.720.1015.1219.1416
08.29.420.515.1011.1214.1317
r 05.29.313.46.812.1018.1415
18.216.411.519.714.1017.1320
2nd.
1st


*6363332
*66323333

old

10



γ, α^n $\bar{\gamma}$ α^2

~~∞1.20 36 48 59 7X~~

$\theta_n = \frac{\iota\beta\bar{\alpha}}{} = \delta\alpha^n\gamma$ $\quad t \to \frac{-1}{t\infty} = \frac{-1}{\infty - t}$

θ_1 ∞∞X5369721.4.8

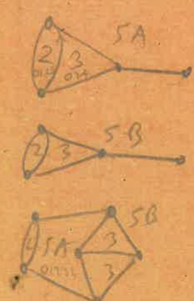
θ_2 : ∞054631X7982

θ_3 : ∞073.1564.2X89

θ_4 : ∞083X25174.6.9

θ_5 : ∞0276184X935

X Y Z



(10)$_5$

(10)$_5$

(18)$_3$

ι = 02.13
β = 024
α = 01.23
θ = 04123

α 13.24
β 01234
ι = 14.23
θ = 031.2.4

A	400
B	432
C	384
D	400
E	432
F	150
G	242
G'	242
H	216
I	150

Quilts [redacted] for $L_2(11)$.

This page contains all webs with an 11-face.
It is closed under "decoration".

		τ^2/δ
γ_0	∞0.1X.25.37.48.69	0
γ_1	∞10.26X.385.497	1
γ_2	∞2754X39860.1	4
γ_3	∞3X40.12876.5.9	9
γ_4	∞4130.29X56.78	5
γ_5	∞57890.142X63	3
γ_6	∞64320.158X79	3
γ_7	∞7X80.16592.3.4	5
γ_8	∞8170.2.345X9.6	9
γ_9	∞9467182350.X	4
γ_X	∞X0.195.247.368	1

So τ^2/δ = 0 1 4 9 5 3
⇒ orders = 2 3 11 5A 5B 6
So we distinguish between 5-els by:-
5A: $\delta = 1$, $\tau = \pm 3$ $\tau^2/\delta = 9$
5B: $\delta = 1$, $\tau = \pm 4$ $\tau^2/\delta = 5$

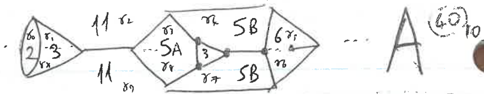
A $2^1 3^2 5^1 \bar{5}^2\, 6\, 11$
D $2^1 3^2 5^2 \bar{5}^1\, 6\, 11$] 5-conj?

B ② $^1 2^1 3^2 5^1 \bar{5}^1 6^1\, 11$
E ② $^1 2^1 2^2 5^1 \bar{5}^1 6^1\, 11$] 5-conj?

C $2^3 3^3 5^3 \bar{5}^3 6^3 11^3$ ⊃ 5-conj?

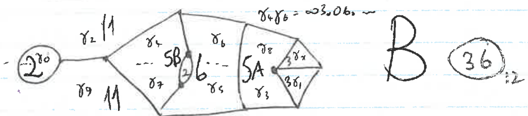
ODD CASES

$\gamma_3\gamma_4 = \infty 0 4 \ldots$ θ: ∞0X5369721.4.8

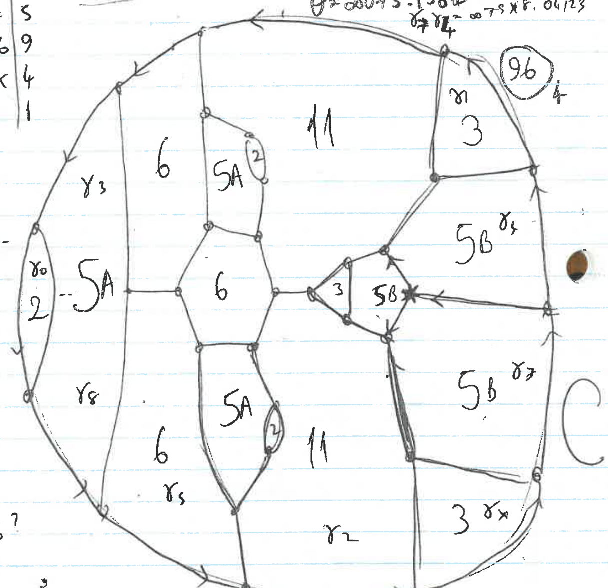


θ: ∞005463 1X7982

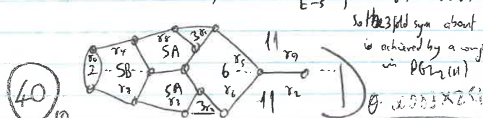
$\gamma_4\gamma_6 = \infty 3.06 \ldots$

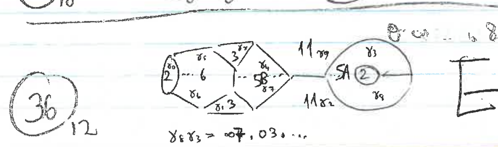


θ = ∞073.1564.2X89
$\gamma_7\gamma_4 = \infty 75X8.04123$



∞3681X497025, i.e. $t \to \frac{3t+1}{t-5}$, takes $\gamma_7 \to \gamma_7\gamma_4 \to \gamma_4$
So the 3 fold sym about * is achieved by a conjugacy in $PGL_2(11)$





$\gamma_8\gamma_3 = \infty 7.03 \ldots$

11-3 friday

$\infty 587X.01342.6.9$ β

$\infty 49.087.13X.265$ α

$\iota = \infty 2.45.69.0X.38.17$

$\iota\beta\bar{\alpha} = \infty 7138.09264.5.X$

$\infty 75X8.04123.6.9 \to \beta'$

$\iota\beta'\bar{\alpha} = \infty 164392875X.0$

$\alpha = \infty 27411.09X635$

$\beta = 13956.267X8.\infty.0$

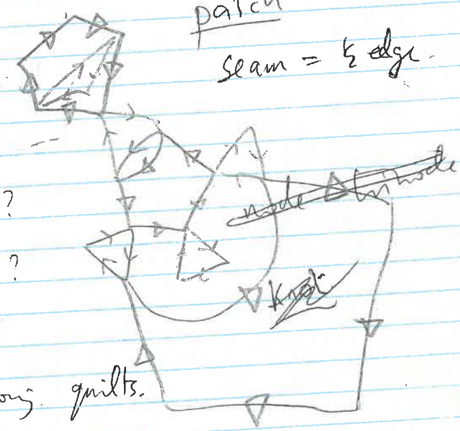
$\iota = \infty 0.69.1X.48.73.25$

$\iota\beta\bar{\alpha} = \infty 51X84.063927$

$\iota\beta^2\bar{\alpha} = \infty 5241.0673X.8.9$

patch

seam = ξ edge



seamlength ?

sewing # ?

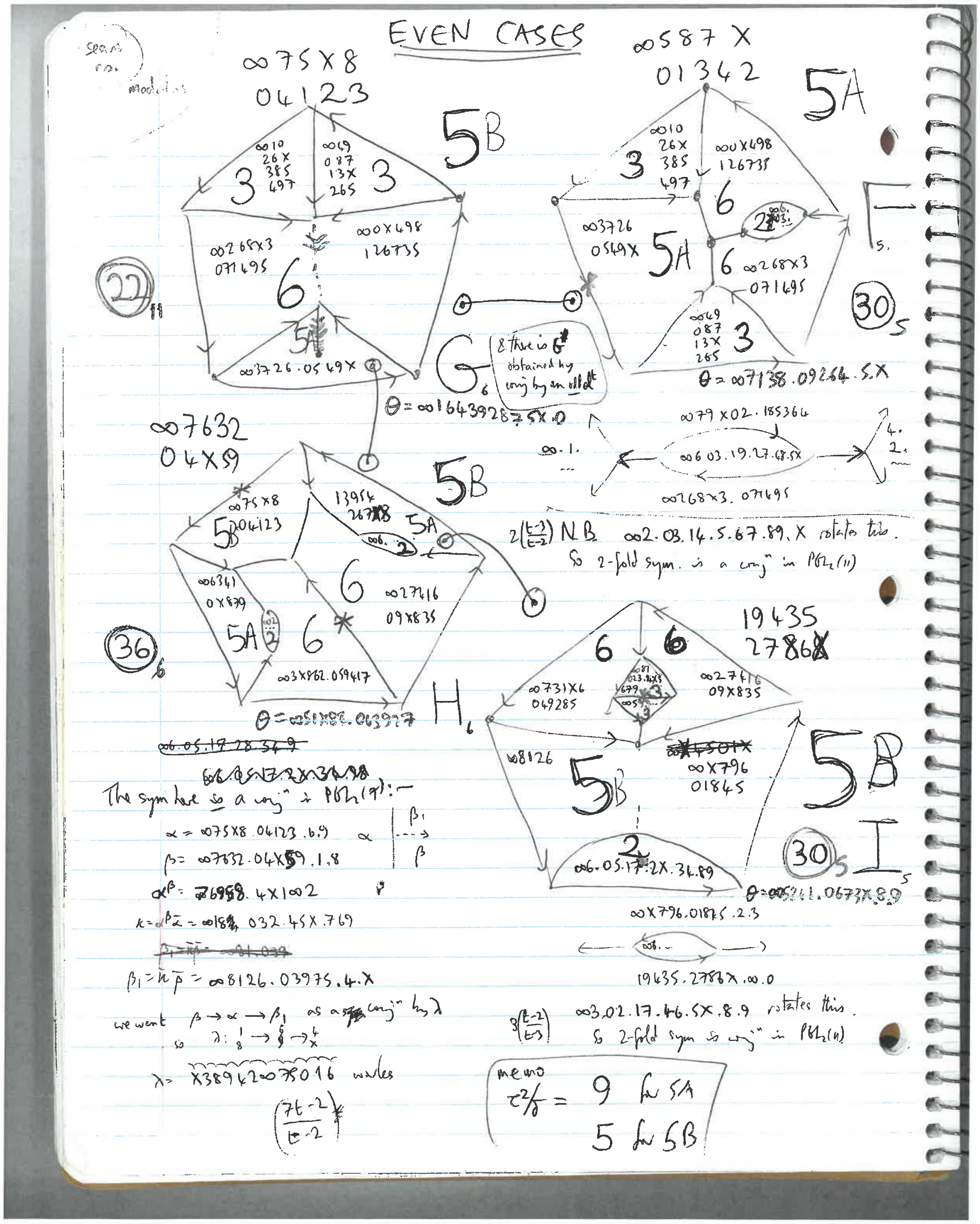
Seam #

F & I are alg. conj quilts.

modulus = order of θ modulo the centre.

5
3|3
5

⟨a,b | [illegible]⟩


EVEN CASES
∞75X8
04123
5B
∞587X
01342
5A
6
3
∞7632
04X9
19435
2786X
The sym here is a conj^n in PGL2(7):—
S. 2-fold sym. is a conj^n in PGL2(11)
memo
9 for 5A
5 for 5B

660	12	6	5	5	6	11	11	
1A	2A	3A	5A	5B	6A	11A	11B	Fus
1	1	1	1	1	1	1	1	:
5	1	-1	0	0	1	b11	**	
5	1	-1	0	0	1	**	b11	
10	2	1	0	0	-1	-1	-1	:
10	-2	1	0	0	1	-1	-1	:
11	-1	-1	1	1	-1	0	0	:
12	0	0	b5	*	0	1	1	:
12	0	0	*	b5	0	1	1	:

$b5 = \frac{-1+\sqrt{5}}{2}$

(3, 5A, 6)

(3, 5B, 6)

255 $1 - 1/11 = \frac{10}{11} \cdot \frac{660}{12 \cdot 5 \cdot 5} = 2$ } EI ×2
* 255′ ⟶ 2 } ×1

256 $1 + 1/11 = \frac{12}{11} \cdot \frac{660}{12 \cdot 5 \cdot 6} = 2$ } BH ×1

266 $1 + 1/5 + 1/5 + 2/10 - 2/10 - 1/11 \to 2$ } F ×2

* 335 $1 + 1/11 = \frac{12}{11} \cdot \frac{660^2}{6 \cdot 6 \cdot 5} = 4$ } BG ×2

336 $1 + \frac{1}{5} + \frac{1}{5} - \frac{1}{10} + \frac{1}{10} - \frac{1}{11} = \frac{72}{55} \cdot \frac{660^2}{666} = 4$ } GI ×2

* 355 $1 - 1/11 = \frac{10}{11} \cdot \frac{660^2}{6 \cdot 5 \cdot 5} \Rightarrow 4 = 4$ } AAAF ×2
* 355′ ⟶ 4 } ×1

356 $1 + \frac{1}{11} = \frac{12}{11} \cdot \frac{660^2}{6 \cdot 5 \cdot 6} \longrightarrow 4$ } DEFG ×1

§ 366 $1 + [illegible] + \frac{1}{10} + \frac{1}{10} - \frac{1}{11} = \frac{39}{55} \cdot \frac{660^2}{666} \to 2\frac{1}{6}$ } I ×2

* 555 $1 + \frac{1}{11} + \frac{b5^3 \bar{b5}}{12} = \frac{25}{33} \cdot \frac{660^4}{555} = 4$ } CDDDGHHH ×2 ?
† 555′ $1 + \frac{1}{11} + \frac{b5^2 b5^* \bar{b5}}{12} = \frac{155}{132} \cdot \frac{660}{555} = 6\frac{1}{5}$ }

556 $1 - \frac{1}{11} = \frac{10}{11} \cdot \frac{660^2}{556} = 4$ } ADGGGHHI ×2
55′6 ⟶ 4 } ×1

566 $1 + \frac{1}{11} = \frac{12}{11} \cdot \frac{660^2}{566} \Rightarrow 4$ } CFHI ×2

666 $1 + \frac{1}{5} + \frac{1}{5} - \frac{1}{10} + \frac{1}{10} - \frac{1}{11} = \frac{72}{55} \cdot \frac{660^2}{666} \to 4$ } AC ×2

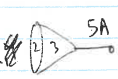
355	AAA F
556	ADGGGHH I
666	A C
256	B H
335	B G
566	C F H I
555	C DDD G HHH
356	D E F G
255	E I
266	F
336	G I
~~566 H I~~	
366	I

$b5^3 = \frac{-1 + 3\sqrt{5} - 3 \cdot 5 + 5\sqrt{5}}{8} = -2 + \sqrt{5}$

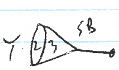
$\therefore \frac{b5^3 \bar{b5}}{12} = \frac{-4}{12} = -1/3$

$b5 \, b5^* = \frac{1-5}{4} = -1$

$\therefore b5^2 b5 \bar{b5} = -b5 \bar{b5} = 1$

X 

355, 555

Y 

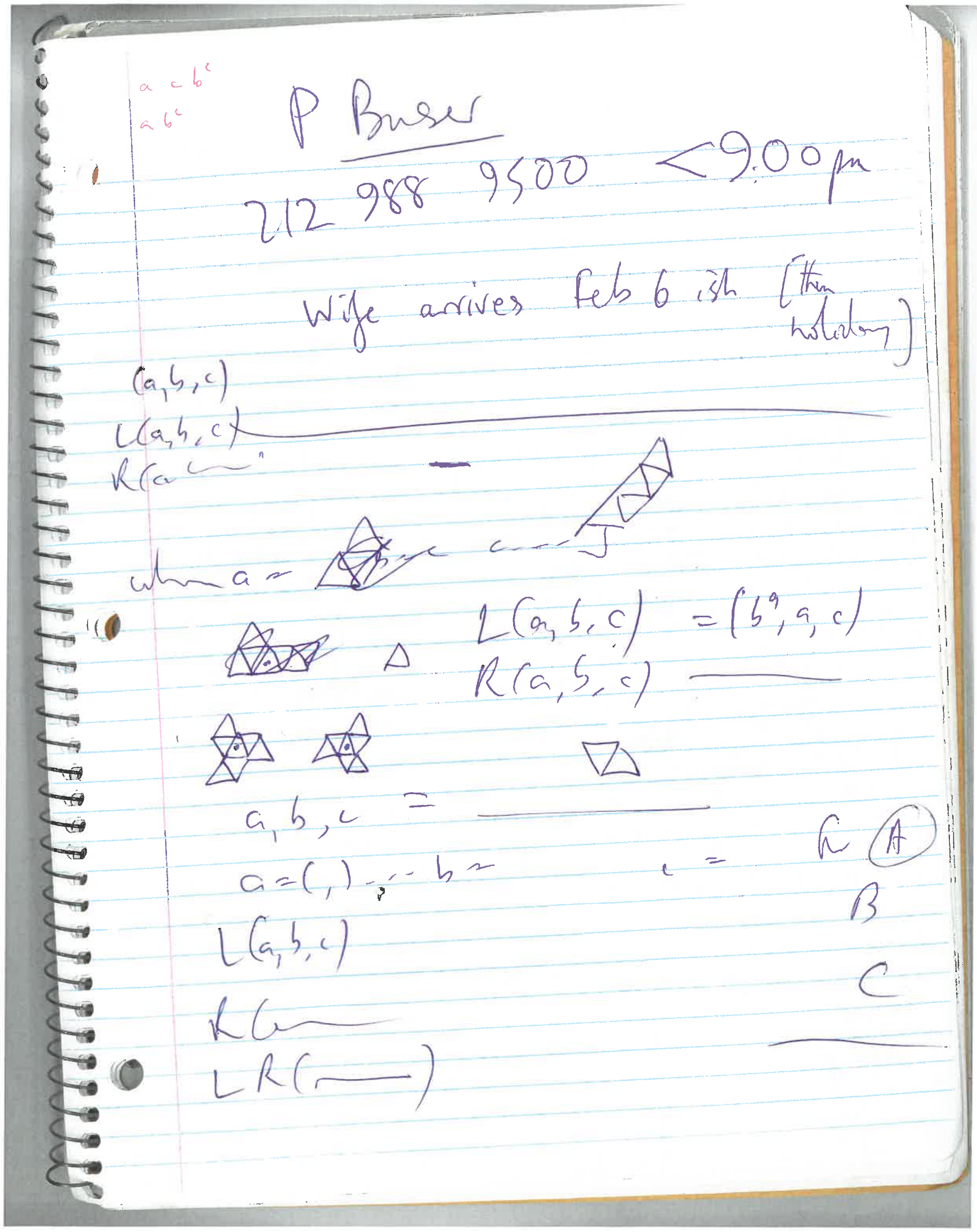
Z 

255′, 355′, 335

*: these happen in A_5

†: in 11:5 & C_5

§: in C_6



a c b^c
a b^c

P Buser

212 988 9500 < 9:00 pm

Wife arrives Feb 6 ish (thru holiday)

(a, b, c)
L(a, b, c)
R(a

when a =

L(a, b, c) = (b^a, a, c)
R(a, b, c)

a, b, c =

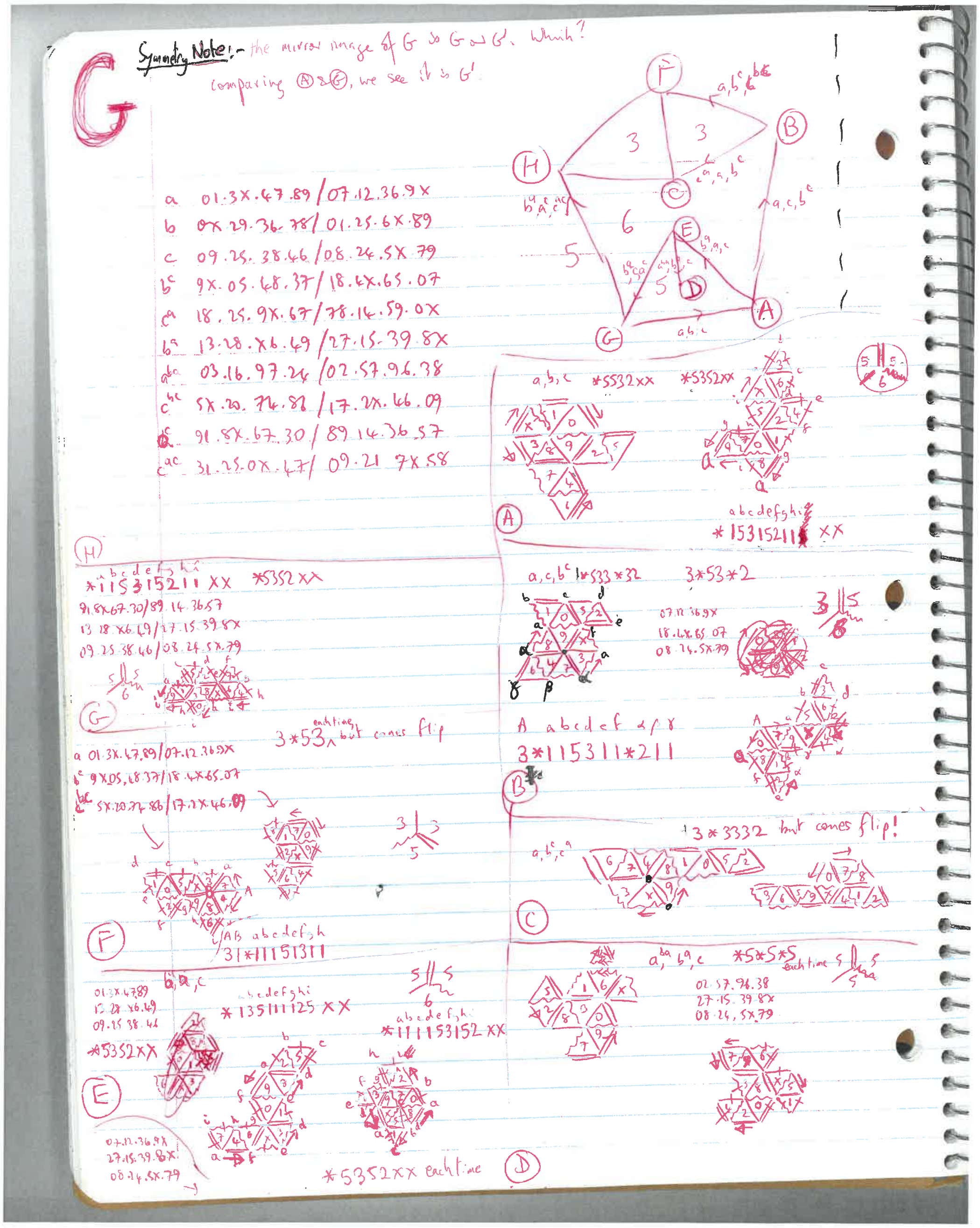
a = (,) ... b = c =

L(a, b, c)

R(

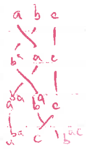
LR( )

A
B
C


G
Symmetry Note: - the mirror image of G is G or G'. Which?
comparing Ⓐ & Ⓖ, we see it is G'.
a 01.3X.47.89/07.12.36.9X
b 0X.29.36.78/01.25.6X.89
c 09.25.38.46/08.24.5X.79
b^c 9X.05.48.37/18.4X.65.07
c^a 18.25.9X.67/78.14.59.0X
b^a 13.28.X6.49/27.15.39.8X
a^ba 03.16.97.24/02.57.96.38
c^bc 5X.20.74.81/17.2X.46.09
c^ac 31.25.0X.47/09.21.7X.58
*5532xx
*5352xx
*153152115 XX
*115315211 XX
*5352 XX
3*53*2
3*53*32
3*115311*211
3*3332 but comes flip!
*5*5*5 each time
31*111151311
*135111125 XX
*111153152 XX
*5352xx
*5352XX each time

F

a 01.3X.47.89 / 07.12.36.9X
b 0X.29.36.78 / 01.25.6X.89
c 05.37.48.9X / 07.18.4X.56
b^a 13.28.6X.49 / 27.15.39.8X
a^{ba} 03.16.97.42 / 02.57.69.38
b^{ac} 17.24.69.8X / 20.68.39.14



> On the symmetry of this examples:

If α, β is on a quilt Q
then $\bar\beta, \bar\alpha$ is on a quilt $\bar{Q}^*$ called ~~conjugate~~ the reflection of Q

If a, b, c are words with $ab = \alpha$, $bc = \beta$

Then c, b, a have $cb = \bar\beta$, $ba = \bar\alpha$

~~[crossed out]~~

So as far as we are concerned, $\bar{Q}^*$ & Q give the same things

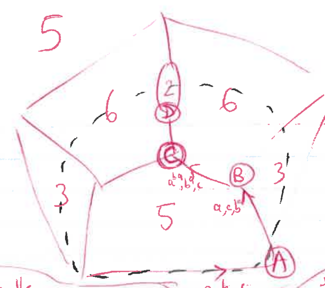
In this case, $\bar{Q}^* = Q$, so mirror image vertices give the same examples as rotates (since the rotation is achieved by duality)

So (A), (B), (C), (D) give all cases.

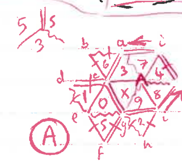
Which is the mirror?

Swapping 1, 3 fixes (A).
Swapping 11, 1 doesn't fix (C). (look at b) in fact it takes (C) to (C)'

So (A) is on mirror, (C) not.



*5533X
3*55X

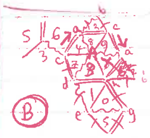


(A)

A abcdefghi
1*131515131X

A abcdefghi
3*1115111151X

01.3X.47.89 / 07.12.36.9X
13.28.6X.49 / 27.15.39.8X
05.37.48.9X / 07.18.4X.56

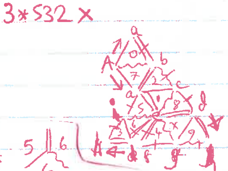


(B)

*53332X
3*532X

A abcdefghi
1*131215313X

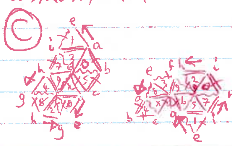
B*1211151 31X



03.16.97.42 / 02.57.69.38
13.28.6X.49 / 27.15.39.8X
05.37.48.9X / 07.18.4X.56

abcd efghi
*1351*31212X
*1531*23121X

(C)

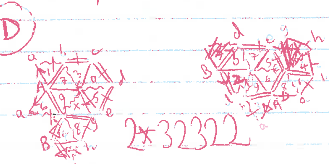


*53*322X

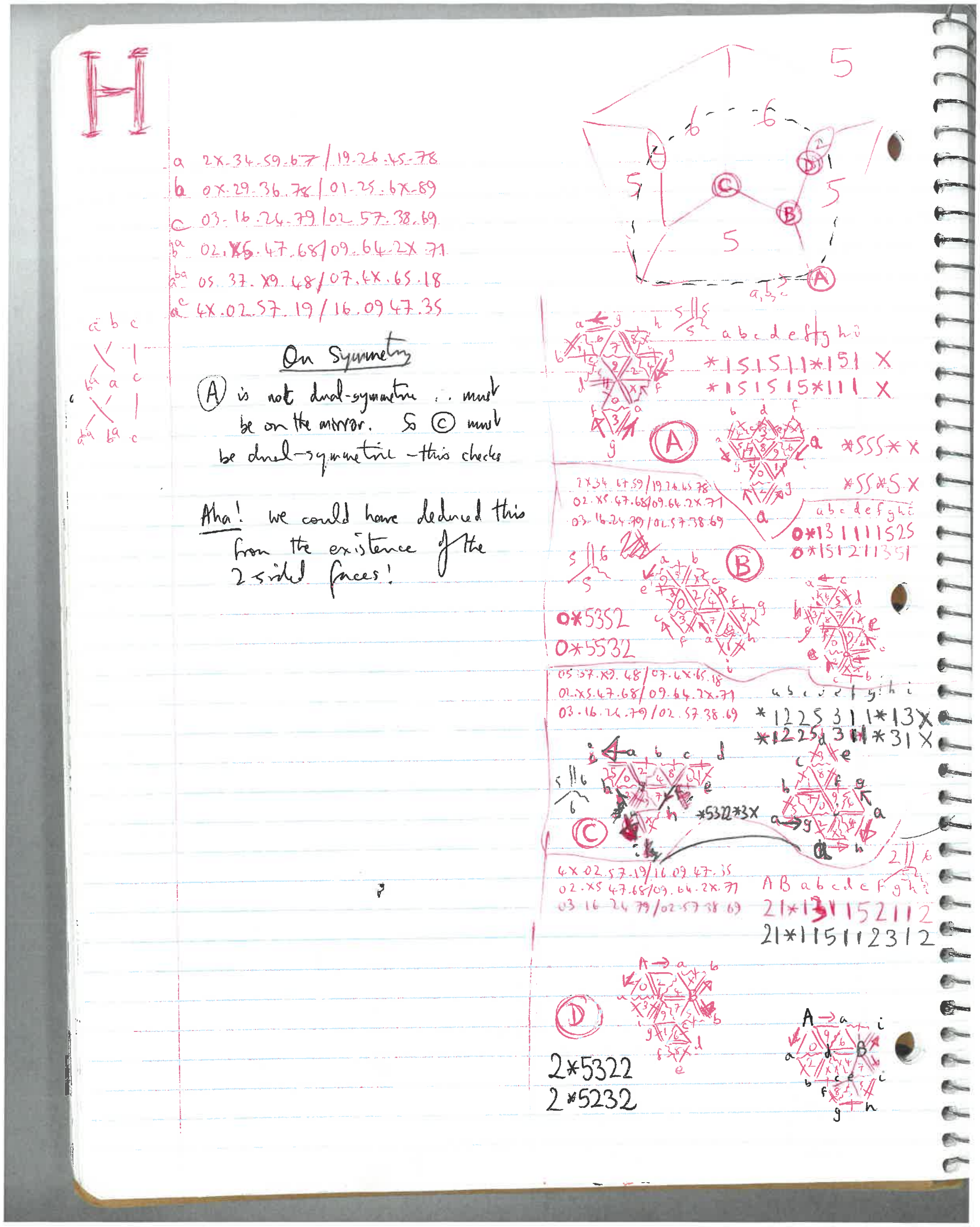
03.16.97.42 / 02.57.69.38
17.24.69.8X / 20.68.39.14
05.37.48.9X / 07.18.4X.56

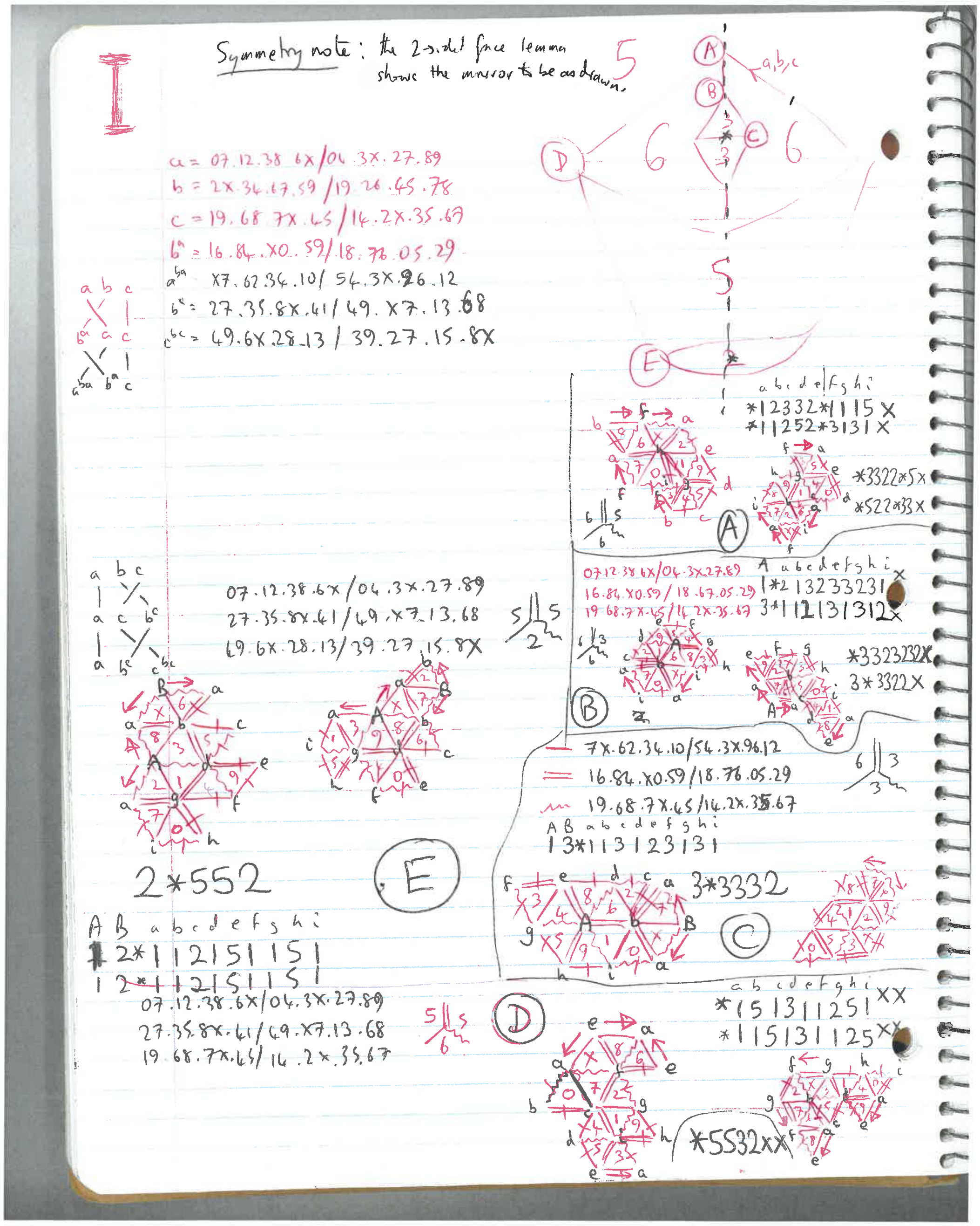
AB abcdefghi
12*321321121
12*231211231

(D)

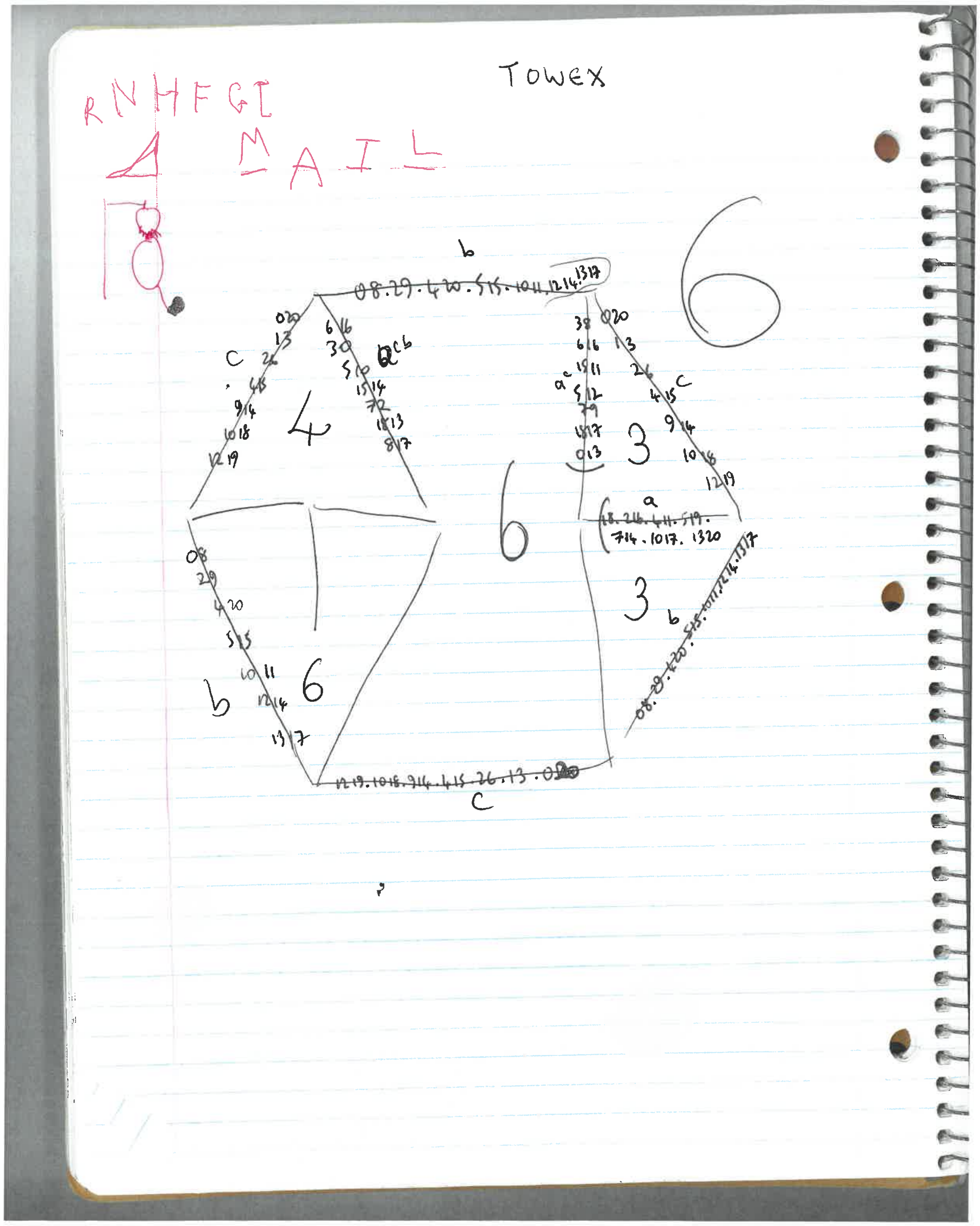


2*32322


H
a 2X.34.59.67 / 19.26.45.78
b 0X.29.36.78 / 01.25.6X.89
c 03.16.24.79 / 02.57.38.69
b^a 02.X5.47.68 / 09.64.2X.71
a^ba 05.37.X9.48 / 07.4X.65.18
a^c 4X.02.57.19 / 16.09.47.35
On Symmetry
(A) is not dual-symmetric ∴ must be on the mirror. So (C) must be dual-symmetric – this checks
Aha! we could have deduced this from the existence of the 2 sided faces!
5
5
5
5
*555*X
*55*5X
0*5352
0*5532
2*5322
2*5232



Symmetry note: the 2-sided face lemma shows the mirror to be as drawn.
a = 07.12.38.6X/04.3X.27.89
b = 2X.34.67.59/19.26.45.78
c = 19.68.7X.45/14.2X.35.67
b^a = 16.84.X0.59/18.76.05.29
a^ba X7.62.34.10/54.3X.96.12
b^c = 27.35.8X.41/49.X7.13.68
c^bc = 49.6X.28.13/39.27.15.8X
a b c d e f g h i
*12332*1115X
*11252*3131X
*3322*5X
*522*33X
A
07.12.38.6X/04.3X.27.89
16.84.X0.59/18.76.05.29
19.68.7X.45/14.2X.35.67
A a b c d e f g h i
1*2 13233231X
3*1 1213131 2X
*3323232X
3*3322X
B
07.12.38.6X/04.3X.27.89
27.35.8X.41/49.X7.13.68
19.6X.28.13/39.27.15.8X
7X.62.34.10/54.3X.96.12
16.84.X0.59/18.76.05.29
19.68.7X.45/14.2X.35.67
A B a b c d e f g h i
13*113123131
3*3332
C
2*552
E
A B a b c d e f g h i
2*1121511151
12*1121511151
07.12.38.6X/04.3X.27.89
27.35.8X.41/49.X7.13.68
19.68.7X.45/14.2X.35.67
D
a b c d e f g h i
*151311251XX
*115131125XX
*5532XX


TOWEX
RNHFGI
MAIL

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