The expected number of rising sequences after a shuffle

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Brad Mann found the following simple expression for the expected number of rising sequences in an n-card deck after an a-shuffle:

$$R_{a,n} = a - \frac{n+1}{a^n} \sum_{r=0}^{a-1} r^n.$$

Brad's derivation involved lengthy gymnastics with binomial coefficients. Obviously this beautiful formula cries out for a one-line derivation, but I still don't see how to do this. The following is the best I have been able to manage.

We look at things from the point of view of doing an a-unshuffle. You get a new rising sequence each time the last occurrence of label i comes after the first occurrence of label $i+1$. More generally, you get a new rising sequence each time the last i comes after the first $i+k$, provided that $i+1, \ldots, i+k-1$ don't occur. The number of labelings with this property is

$$(a-k+1)^n - (a-k)^n - n(a-k)^{n-1}$$

(From all labelings omitting $i+1, \ldots, i+k-1$ discard those that omit i, and then those where there is some card labeled i (n possibilities for this card) such that no card that comes before it is labelled $i+k$ and no card after it is labeled i.) For any specified value of k there are $a-k$ possibilities for i, so

$$\begin{aligned} R_{a,n} &= 1 + \frac{1}{a^n} \sum_{k=1}^{a} (a-k) \left[(a-k+1)^n - (a-k)^n - n(a-k)^{n-1} \right] \\ &= 1 + \frac{1}{a^n} \sum_{s=0}^{a-1} s \left[(s+1)^n - (s^n + ns^{n-1}) \right]. \end{aligned}$$

When a is large,

$$\begin{aligned} R_{a,n} &\approx 1 + \frac{1}{a^n} \sum_{s=0}^{a-1} s \binom{n}{2} s^{n-2} \\ &= 1 + \frac{1}{a^n} \binom{n}{2} \sum_{s=0}^{a-1} s^{n-1} \\ &\approx 1 + \frac{1}{a^n} \binom{n}{2} \frac{a^n}{n} \\ &= \frac{n+1}{2}, \end{aligned}$$

which is the expected number of rising sequences in a perfectly shuffled deck.

A little juggling is required to transform the expression for $R_{a,n}$ derived above into the form that Brad gave. As I said before, I do not see how to write down Brad's form directly.