

# Riffles, ruffles, and the turning algebra

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## Abstract

The *rising algebra* is a subalgebra of the group algebra of the symmetric group  $S_n$ , gotten by lumping together permutations having the same number of rising sequences. This well-known algebra arises naturally when studying riffle shuffles. Here we introduce a number of other subalgebras that arise naturally when stuffing ‘ruffles’, which are like riffles except that after cutting the deck you turn over the bunch of cards that were on the bottom.

This orphaned draft offers no context or motivation, and uses idiosyncratic notation and terminology that ‘seemed like a good idea at the time’. We’re making it available because it has been cited in this form.

## 1 To and fro

### 1.1 Natural order

The theory of shuffling grows out of Jim Reeds’s fundamental observation that to understand the riffle shuffle, you have to look at it backwards. Now, keeping straight the difference between  $\sigma$  and  $\sigma^{-1}$  is a chore whenever you deal with permutations; having to try to keep everything *backwards* is pretty

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near impossible (for us, at least). To give ourselves a fighting chance, we have to write composition of functions in the natural, left-to-right order.

**WARNING.** Throughout this paper we will compose functions in natural order:

$$(\sigma\tau)[x] = \tau[\sigma[x]].$$

To try to minimize confusion, we will use superscripts whenever possible, so that

$$x^{\sigma\tau} = (x^\sigma)^\tau.$$

We will also have occasion to use Wolfram's postfix notation, so that

$$(x//\sigma)//\tau = x//(\sigma\tau).$$

## 1.2 The permutation group

Let  $S_n$  denote the group of bijections from  $\{1, \dots, n\}$  to itself, with functions composed in natural order:

$$i^{\sigma\tau} = (i^\sigma)^\tau.$$

As with any function defined on  $\{1, \dots, n\}$ , we can represent a permutation  $\sigma$  as an  $n$ -tuple.

$$(1^\sigma, \dots, n^\sigma).$$

We will adopt a variant of the cycle notation for permutations, using  $\langle i j \rangle$  to denote the transposition switching  $i$  and  $j$ , and letting

$$\langle i_1 \dots i_n \rangle = \langle i_{n-1} i_n \rangle \langle i_1 \dots i_{n-1} \rangle.$$

(Don't forget: natural order!)

For example, if  $n = 3$ , and  $a, b$  are the standard 'braid' generators

$$a = \langle 1 2 \rangle = (2, 1, 3),$$

$$b = \langle 2 3 \rangle = (1, 3, 2),$$

then

$$ab = \langle 1 2 \rangle \langle 2 3 \rangle = \langle 1 3 2 \rangle = (3, 1, 2).$$

This example demonstrates what appears to be a serious drawback of the  $n$ -tuple representation, for while this representation 'tells us where they

went', it doesn't *show* us. To be more specific, it seems very natural to represent the effects of  $a$ ,  $b$ , and  $ab$  by drawing before-and-after diagrams:

$$a = \frac{(1, 2, 3)}{(2, 1, 3)},$$

$$b = \frac{(1, 2, 3)}{(1, 3, 2)},$$

$$ab = \frac{(1, 2, 3)}{(2, 3, 1)}.$$

Here the numerator  $(1, 2, 3)$  isn't conveying much information, but if we omit it, the  $n$ -tuple that remains is the representation, not of the permutation we're looking at, but of its inverse. So it seems like the  $n$ -tuple representation of a permutation is just backwards from what we want. Where did we go wrong?

Where we went wrong, of course, was in discarding the 'numerator' of our before-and-after diagram, which we should properly think of as a fraction. If we interpret the fraction  $\frac{\sigma}{\tau}$  to mean  $\sigma\tau^{-1}$ , then everything is groovy:

$$a = \frac{(1, 2, 3)}{(2, 1, 3)} = (2, 1, 3),$$

$$b = \frac{(1, 2, 3)}{(1, 3, 2)} = (1, 3, 2),$$

$$ab = \frac{(1, 2, 3)}{(2, 3, 1)} = (3, 1, 2).$$

Moreover, if we rewrite  $b$  as

$$b = \frac{(1, 2, 3)}{(1, 3, 2)} = \frac{(2, 1, 3)}{(2, 3, 1)},$$

then we get the very natural 'braidlike' equations

$$a = \frac{(1, 2, 3)}{(2, 1, 3)},$$

$$b = \frac{(2, 1, 3)}{(2, 3, 1)},$$

$$ab = \frac{(1, 2, 3)}{(2, 3, 1)}.$$

Of course we will have to be careful to remember that in the formula

$$\frac{\sigma}{\tau} = \sigma\tau^{-1},$$

the  $\tau^{-1}$  comes after (i.e., to the *right* of) the  $\sigma$ . This is actually very natural, if you consider that  $\frac{\sigma}{\tau}$  is pronounced ‘sigma divided by tau’.

This whole question of ‘Which came first, the sigma or the tau?’ disappears if we represent our  $n$ -tuples as column vectors, and transpose the fractions  $\frac{\sigma}{\tau}$  to  $\sigma|\tau$ . Doing this yields the very congenial equation

$$ab = \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left| \left( \begin{array}{c} 2 \\ 1 \\ 3 \end{array} \right) \left( \begin{array}{c} 2 \\ 1 \\ 3 \end{array} \right) \right| \left( \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \right) = \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left| \left( \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \right) \right| \left( \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \right),$$

which we abbreviate to

$$ab = \left( \begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{array} \right) \left( \begin{array}{c|c} 2 & 2 \\ 1 & 3 \\ 3 & 1 \end{array} \right) = \left( \begin{array}{c|c} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{array} \right).$$

This vertical representation of permutations is particularly appropriate in discussing shuffling, since it makes it easy to visualize the cards as they appear in the deck.

The conventions and notations that we have adopted fit in well with the representation of permutations as matrices. Here we take our cue from the theory of Markov chains, where a probability distribution is most conveniently represented as a row vector  $(p_1, \dots, p_n)$  of positive numbers summing to 1. A permutation  $\sigma$  corresponds naturally to the Markov transition matrix `permmatrix` $[\sigma]$ , where

$$\text{permmatrix}[\sigma]_{ij} = \delta_{\sigma[i],j}.$$

Multiplying our row vector  $(p_1, \dots, p_n)$  by `permmatrix` $[\sigma]$  (on the right!) yields

$$(p_1, \dots, p_n) \text{permmatrix}[\sigma] = (p_{\sigma^{-1}[1]}, \dots, p_{\sigma^{-1}[n]}),$$

which fortunately turns out to be the effect of taking the quantities  $p_i$  and moving them from their initial position  $i$  to position  $\sigma[i]$ . Holding our breath, we check, and to our delight we find that, with  $a$  and  $b$  as above,

$$\text{permmatrix}[a] \text{permmatrix}[b]$$

$$\begin{aligned}
&= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \text{permmatrix}[ab]
\end{aligned}$$

All's well with the world!

## 2 Actions and reactions

Let  $G$  be a group and  $M$  a monoid. (Actually, this whole discussion might go through when  $G$  is only a monoid, but we prefer to assume  $G$  is a group—if only for alphabetical reasons—until there is some good reason for generalizing to a monoid.)

Let  $G$  act on  $M$  on the right by automorphisms, so that

$$(m^g)^h = m^{gh}$$

and

$$(mn)^g = m^g n^g.$$

When we need to refer to this action by name, we will attach this name to the associated homomorphism  $\rho : G \rightarrow \text{Aut}[M]$ . Here  $\text{Aut}[M]$  denotes the group of automorphisms with natural-order composition, and  $\rho$  is our default name for group actions.

Given an action  $\rho$  of  $G$  on  $M$ , the *semidirect product*  $G \times_\rho M$  is the monoid consisting of the set  $G \times M$  together with the composition law

$$(g, m)(h, n) = (gh, m^h n).$$

To check the associative law, we note that

$$((g, m)(h, n))(i, o) = (g, m)((h, n)(i, o)) = (ghi, m^{hi} n^i o).$$

$G$  is isomorphic to the submonoid  $G \times \{1\}$  of  $G \times_\rho M$ . The map  $(g, m) \mapsto (g, 1)$  is a monoid-homomorphism onto this submonoid; its kernel is the ‘normal’ submonoid  $\{1\} \times M$ , which is isomorphic to  $M$ .

Given an action  $\rho$  of  $G$  on  $M$ , a map  $\gamma : M \rightarrow G$  is called a *reaction to  $\rho$*  if

$$\gamma[m]\gamma[n] = \gamma[m^{\gamma[n]}n],$$

i.e., if

$$(\gamma[m], m)(\gamma[n], n) = (\gamma[m]\gamma[n], m^{\gamma[n]}n) = (\gamma[m^{\gamma[n]}n], m^{\gamma[n]}n).$$

This means that the set

$$\{(g, m) : g = \gamma[m]\}$$

is a submonoid of  $G \times M$ . As a set, the elements of this submonoid correspond naturally to the elements of  $M$ , only the product in  $M$  has been twisted through the interaction with  $G$ . We denote this new product by  $*_\gamma$  (leaving  $\rho$  to be inferred from context), so that

$$m *_\gamma n = m^{\gamma[n]}n,$$

and we denote the monoid  $M$  with product  $*_\gamma$  by

$$G \times_\gamma M.$$

If we ever have to call this something, we will call it the *demisemidirect product* of  $G$  and  $M$  with respect to  $\rho$  and  $\gamma$ .

## 3 Ruffles and ruffles

### 3.1 The radix monoid

Let  $n$  be a positive integer, e.g. 52. Denote by  $\text{Radix}_n$  the monoid with elements  $(a, (x_1, \dots, x_n)) : a \geq 1, 0 \leq x_i < n$ , and multiplication

$$(a, (x_1, \dots, x_n))(b, (y_1, \dots, y_n)) = (ab, (bx_1 + y_1, \dots, bx_n + y_n)),$$

or in simplified notation,

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_a \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_b = \begin{pmatrix} bx_1 + y_1 \\ \vdots \\ bx_n + y_n \end{pmatrix}_{ab}.$$

We think of the elements of  $\text{Radix}_n$  as lists of digits in the specified radix  $a$ ; we combine two lists entry-by-entry (in the natural order!), interpreting the product of  $\text{rad}(x, a)$  and  $\text{rad}(y, b)$  as  $\text{rad}((x, y), (a, b))$ , a two-digit number in the hybrid radix  $(a, b)$ , where the first digit  $x$  is in radix  $a$  and the second digit  $y$  in radix  $b$ .

Note that if we represent  $\text{rad}(x, a)$  as the linear polynomial  $aX + x$ , then the mixed-radix product of  $\text{rad}(x, a)$  and  $\text{rad}(y, b)$  corresponds to the composition (in natural order) of the corresponding linear functions:

$$(X \mapsto aX + x)(X \mapsto bX + y) = (X \mapsto abX + bx + y).$$

### 3.2 The riffle monoid

Now let  $S_n$  act on  $\text{Radix}_n$  by permuting the list entries, and let  $\text{Radix}_n$  react via the function `riffle` by interpreting the entries in the list of digits base  $a$  as portraying the effect of an  $a$ -handed riffle:

$$\begin{aligned} & \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_2 // \text{riffle} = \left( \begin{array}{c|c} 1 & 3 \\ 2 & 4 \\ 3 & 1 \\ 4 & 5 \\ 5 & 2 \end{array} \right), \\ & \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}_3 * \text{riffle} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}_3 * \text{rad} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 3 \\ 4 \end{pmatrix}_6, \\ & \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}_3 // \text{riffle} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_2 // \text{riffle} \\ & = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 2 \\ 4 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 1 \\ 4 & 5 \\ 5 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \left( \begin{array}{c|c} 1 & 4 \\ 2 & 5 \\ 3 & 2 \\ 4 & 1 \\ 5 & 3 \end{array} \right) \left( \begin{array}{c|c} 4 & 2 \\ 5 & 1 \\ 2 & 4 \\ 1 & 3 \\ 3 & 5 \end{array} \right) \\
&= \left( \begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \\ 5 & 5 \end{array} \right) \\
&= \left( \begin{array}{c} 3 \\ 1 \\ 4 \\ 3 \\ 4 \end{array} \right) //\text{riffle}.
\end{aligned}$$

We call the monoid arising from this reaction the *riffle monoid*:

$$\text{Riffle}_n = S_n \times_{\text{riffle}} \text{Radix}_n.$$

### 3.3 The Gray monoid

As a variation on  $\text{Radix}_n$ , we introduce  $\text{Gray}_n$ , which is to  $\text{Radix}_n$  as the Gray code is to binary. Specifically,  $\text{Gray}_n$  has the same elements as  $\text{Radix}_n$ , but the new multiplication

$$\left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right)_a *_{\text{gray}} \left( \begin{array}{c} y_1 \\ \vdots \\ y_n \end{array} \right)_b = \left( \begin{array}{c} bx_1 + (x_1 \text{ even? } y_1 : b - 1 - y_1) \\ \vdots \\ bx_n + (x_n \text{ even? } y_n : b - 1 - y_n) \end{array} \right)_{ab}.$$

Here we combine the Gray digits  $\text{gray}(x, a)$  and  $\text{gray}(y, b)$  by treating  $(x, y)$  as a two-digit number in the hybrid Gray base  $(a, b)$ , where the lower order Gray digit runs alternately up and down, so that for example counting in Gray base  $(3, 2)$  goes

$$(0, 0), (0, 1), (1, 1), (1, 0), (2, 0), (2, 1).$$



### 3.4 The ruffle monoid

To describe up-down riffles, or *ruffles*, we use the monoid  $\mathbf{Ruffle}_n$ , which we get by letting  $S_n$  act as usual on  $\mathbf{Gray}_n$ , and letting  $\mathbf{Gray}_n$  react via the function  $\mathbf{ruffle}$  by interpreting the entries in the list of digits as portraying the effect of an  $a$ -handed ruffle:

$$\begin{aligned} & \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_2 // \mathbf{ruffle} = \begin{pmatrix} 1 & | & 5 \\ 2 & | & 4 \\ 3 & | & 1 \\ 4 & | & 3 \\ 5 & | & 2 \end{pmatrix}, \\ & \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}_3 * \mathbf{ruffle} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}_3 * \mathbf{gray} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}_6, \\ & \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}_3 // \mathbf{ruffle} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_2 // \mathbf{ruffle} \\ & = \begin{pmatrix} 1 & | & 4 \\ 2 & | & 3 \\ 3 & | & 5 \\ 4 & | & 1 \\ 5 & | & 2 \end{pmatrix} \begin{pmatrix} 1 & | & 5 \\ 2 & | & 4 \\ 3 & | & 1 \\ 4 & | & 3 \\ 5 & | & 2 \end{pmatrix} \\ & = \begin{pmatrix} 1 & | & 4 \\ 2 & | & 3 \\ 3 & | & 5 \\ 4 & | & 1 \\ 5 & | & 2 \end{pmatrix} \begin{pmatrix} 4 & | & 2 \\ 3 & | & 1 \\ 5 & | & 4 \\ 1 & | & 5 \\ 2 & | & 3 \end{pmatrix} \\ & = \begin{pmatrix} 1 & | & 2 \\ 2 & | & 1 \\ 3 & | & 4 \\ 4 & | & 5 \\ 5 & | & 3 \end{pmatrix} \end{aligned}$$

$$= \left( \begin{array}{c} 2 \\ 1 \\ 3 \\ 5 \\ 3 \end{array} \right) // \text{ruffle}.$$

This reaction yields the *ruffle monoid*:

$$\text{Ruffle}_n = S_n \times_{\text{ruffle}} \text{Gray}_n.$$

## 4 New algebras from old

### 4.1 Lumped monoids

A function  $\mu : M \rightarrow S$  from the monoid  $M$  to an arbitrary set  $S$  determines the equivalence relation  $\equiv_\mu$ , where  $a \equiv_\mu b$  if and only if  $\mu[a] = \mu[b]$ . We say that the function  $\mu$  is a *lumping* if (the characteristic functions of) the  $\mu$ -equivalence classes constitute a basis for a subalgebra of the monoid algebra  $\mathbf{Q}[M]$  (or  $\mathbf{C}[M]$ , if you prefer). (See Pitman [?].) Combinatorially, this amounts to requiring that the  $\mu$ -equivalence classes  $[a]$  all be finite, and that there exist *structure constants*  $C_{[a],[b],[c]}$  such that for any  $a, b, c \in M$  there are exactly  $C_{[a],[b],[c]}$  ways of writing  $xy = c$  with  $x \in [a]$ ,  $y \in [b]$ .

### 4.2 Do the right thing

Let  $M$  and  $N$  be monoids,  $\mu$  a lumping of  $M$ , and  $\nu$  a function on  $N$  that we hope to show is a lumping. We say that a homomorphism  $f : M \rightarrow N$  *does the right thing* if in the monoid algebra  $\mathbf{Q}[N]$  the elements  $\sum_{x \in [a]_\mu} f(x)$  belong to and span the subspace spanned by (the characteristic functions of) the  $\nu$ -equivalence classes. Combinatorially, this amounts to requiring that there exist a matrix  $D = \{D_{[a]_\mu, [b]_\nu}\}$  of what we might call *restructure constants*, such that for any  $a \in M$ ,  $b \in N$  there are exactly  $D_{[a]_\mu, [b]_\nu}$  ways of writing  $f(x) = b$  with  $x \in [a]_\mu$ ; in addition, the row-space of the matrix  $D$  must contain the standard basis vectors, a requirement that in our examples will follow from the fact that we can order the rows and columns of the matrix  $D$  so that it becomes lower-triangular, with non-zero entries on the diagonal.

**Theorem.** If  $f : M \rightarrow N$  does the right thing with respect to a lumping  $\mu$  of  $M$  and a function  $\nu$  on  $N$  then  $\nu$  is a lumping of  $N$ . ♣

## 5 Shuffling and its algebras

### 5.1 Hand-equivalence and cut-equivalence

In the monoids  $\mathbf{Riffle}_n$ , and  $\mathbf{Ruffle}_n$  we can lump elements together according to the value of the radix  $a$ . Let's call the resulting equivalence relation **hand**-equivalence, since we are lumping together shuffles involving the same number of hands. Note that the subalgebra yielded by the lumping **hand** is commutative: Indeed, it is isomorphic to the monoid algebra of the natural numbers, because every  $ab$  riffle arises in one and only one way as an  $a$ -riffle followed by a  $b$ -riffle (or vice versa).

Alternatively, we can refuse to identify two lists unless in addition to sharing the same radix  $a$ , each base  $a$  digit occurs the same number of times in the second list as it does in the first. This more discerning equivalence relation we call **cut**-equivalence, since now we are lumping together shuffles only if the cards are cut and distributed among the  $a$  hands in the same way.

### 5.2 Rising sequences

Given a permutation  $\sigma \in S_n$ , we cut the sequence  $1, \dots, n$  into subsequences called the *rising sequences* of  $\sigma$  by dividing it between  $i$  and  $i + 1$  whenever  $\sigma[i + 1] < \sigma[i]$ . The number of rising sequences in  $\sigma$  tells the minimum number of hands you need in order to produce  $\sigma$  as the result of a single riffle, and the specific division into rising sequences tells where you have to make the cuts in order to accomplish this.

The notion of rising sequences suggests two equivalence relations on  $S_n$ . We say that two permutations are **rising**-equivalent if they have the same number of rising sequences, and **risingsequence**-equivalent if in addition the rising sequences of the two permutations are exactly the same.

The map **riffle** does the right thing with respect to **hand** on  $\mathbf{Riffle}_n$  and **rising** on  $S_n$ . To verify this, we must check that the number of ways of realizing a given permutation  $\sigma$  as the result of an  $a$ -handed shuffle depends only on the rising number of  $\sigma$ . This fundamental observation about riffles is due to Bayer and Diaconis [?]; the proof is a standard 'stars-and-bars' argument.

Since **riffle** does the right thing, **rising** is a lumping, and yields a commutative subalgebra of the group algebra of  $S_n$ , which we call the *rising algebra*. (See Bayer and Diaconis [?], Pitman [?].)

The map `riffle` also does the right thing with respect to `cut` and `risingsequence`, thus yielding the larger *rising sequence algebra*. This second algebra is commonly called the ‘descent algebra’ (see Bayer and Diaconis [?], Hanlon [?]; we prefer to call it the ‘rising sequence algebra’ because we feel this is more in line with the general policy:

As you go through life make this your goal,  
Watch the doughnut, not the hole!

### 5.3 Turning points and oriented rising sequences

We say that a permutation has a turning point at  $i$ ,  $1 \leq i < n$ , if the graph of the permutation, extended to map 0 to 0 and interpolated linearly to give a piecewise linear mapping from  $[0, n]$  to itself, has a local maximum or minimum at  $i$ . Note that 1 and  $n$  are treated differently in this definition, in that we call 1 a turning point if 2 ends up coming before 1, but we never call  $n$  a turning point. The more symmetrical notion of ‘reduced’ turning number will be discussed later.

We say that two permutations are **turning-equivalent** if they have the same number of turning points. (The stronger notion requiring that in addition they have exactly the same set of turning points coincides with the `risingsequence` relation, so we can ignore it.)

While the rising number of the identity permutation is 1, the turning number of the identity is 0, and in general the turning number of a permutation is 1 less than we would hope and expect. This anomaly stems from the fact that while rising sequences begin and end at the interstices between consecutive integers in the domain of the permutation, turning points occur at the integers themselves. To get a more felicitous analog of rising sequences, we must look not at permutations, but at *oriented permutations* (also called ‘signed permutations’). In card-shuffling terms, an oriented permutation keeps track of the way the cards are facing as well as their order. We divide the sequence  $1, \dots, n$  into subsequences, called *oriented rising sequences*, according to the cuts we would need to make in order to achieve the specified arrangement in a single riffle with the minimum number of hands. Note that some of these subsequences may have length 0, though no two of them in a row will have length 0. The maximum oriented rising number is  $2n$ : To turn over each card in place with a single riffle, you need  $2n$  hands.

The turning number of a permutation can then be viewed as one less than

the minimum oriented rising number of an oriented permutation that reduces to the given permutation when the orientations of the cards are ignored.

## 5.4 The turning algebra and the oriented rising algebra

The map  $\text{ruffle} : \text{Ruffle}_n \rightarrow S_n$  does the right thing with respect to **hand** and **turning**. As you would expect, the best way to see that  $\text{ruffle}$  does indeed do the right thing is to factor it through the group  $\bar{S}_n$  of oriented permutations. Writing

$$\text{ruffle}[m] = \text{orientedruffle}[m] // \pi_{\bar{S}_n, S_n},$$

we observe first that  $\text{orientedruffle}$  does the right thing with respect to **hand** on  $\text{Ruffle}_n$  and  $\text{orientedrising}$  on  $\bar{S}_n$ . Thus  $\text{orientedrising}$  is a lumping. Then we observe that  $\pi$  does the right thing with respect to  $\text{orientedrising}$  and **turning**, so **turning** is a lumping. Verifying that these two maps do indeed do the right thing involves showing that the number of ways of obtaining a given oriented permutation as the result of an  $a$ -riffle depends only on the oriented rising number, and the number of ways of obtaining a given permutation as the result of an oriented permutation with specified oriented rising number depends only on the turning number. As in the case of riffles, these verifications involve elementary counting arguments.

Thus we obtain a commutative subalgebra of the group algebra of  $S_n$ , the *turning algebra*, by way of a commutative subalgebra of the group algebra of  $\bar{S}_n$ , the *oriented rising algebra*.

## 5.5 The reduced turning number

The *reduced turning number* of a permutation differs from the turning number in that it refuses to recognize a turning point at 1. Thus the identity permutation and the permutation  $\Delta$  that reverses  $1, \dots, n$  (turning over the deck) both have reduced turning number 0.

To treat the reduced turning number with the machinery we have developed, we need a version of riffling where instead of always turning over the odd-numbered piles, we turn over the odds or the evens depending on a specified direction. Thus we replace  $\text{Gray}_n$  with  $\{\text{up}, \text{down}\} \times \text{Gray}_n$ . We extend the action of  $S_n$  by having it leave the direction alone, and we let

$\{\text{up}, \text{down}\} \times \text{Gray}_n$ . react by interpreting the list of digits as the effect of a regular ('up-down') riffle or a reverse ('down-up') riffle according to the value of the direction:

$$\begin{aligned} (\text{up}, (2, (1, 1, 0, 1, 0))) // \text{directedruffle} &= \frac{(1, 2, 3, 4, 5)}{(5, 4, 1, 3, 2)}, \\ (\text{down}, (2, (1, 1, 0, 1, 0))) // \text{directedruffle} &= \frac{(1, 2, 3, 4, 5)}{(3, 4, 2, 5, 1)}. \end{aligned}$$

This reaction yields the *directed riffle monoid*:

$$S_n \times_{\text{directedruffle}} (\{\text{up}, \text{down}\} \times \text{Gray}_n)$$

The map `directedruffle` does the right thing (here again by way of  $\bar{S}_n$ ), and we get the a commutative subalgebra of the group algebra of  $S_n$ , the *reduced turning algebra*, by way of the corresponding commutative subalgebra of the group algebra of  $\bar{S}_n$ .

## 6 Bijective correspondences

The existence of the rising algebra is equivalent to the fact that any two permutations with  $k$  rising sequences arise in the same number of ways as products of permutations with  $i$  and  $j$  rising sequences. Moreover, in this case there is a natural bijection between the sets of factorings of the two permutation (and also between these factorings and the factorings where the roles of  $i$  and  $j$  are reversed). The existence of these bijections follows on general principles from the fact that the matrix of restructure constants associated with the map `riffle`, which induces the rising algebra from the combinatorially trivial `hand`-equivalence subalgebra of the riffle algebra, can be written in lower-triangular form with *1's on the diagonal*. The same goes for the oriented rising algebra. However, in the case of the turning algebra, the diagonal restructure constants are powers of two. Of course, since the sets of factorings in question are the same size, there will of still exist bijections. However, there will no longer be any reason to expect that there will exist any natural bijections (whatever that might mean). Thus it appears that from a combinatorial point of view, the turning algebra is essentially more complicated than the oriented rising algebra (from which it arises by lumping) and the rising algebra.