# Trees and expectations

TSILB*

Version 0.91, 10 March 2003

## Notation for probability trees

Elementary probability theory is hampered by the lack of a good linear notation for probability trees. Here we propose the $\to$ notation, based on representing a probability distribution with outcomes $x_1, \ldots, x_n$ and associated probabilities $p_1, \ldots, p_n$ as

$$\{p_1 \to x_1, \ldots, p_n \to x_n\}.$$

Definition. A *probability tree* is defined recursively as either an outcome (assumed distinguishable as such), or an expression of the form

$$\{p_1 \to T_1, \ldots, p_n \to T_n\}$$

where $T_1, \ldots, T_n$ are probability trees, and $p_1, \ldots, p_n$ are non-negative real numbers adding to 1.

Example: I bet a dollar on the toss of a fair coin:

$$\{\frac{1}{2} \to 1, \frac{1}{2} \to -1\}.$$

Example: I am playing a best-of-3 coin-tossing tournament. The prize is $a$, and I have already won one toss:

$$\{\frac{1}{2} \to a, \frac{1}{2} \to \{\frac{1}{2} \to a, \frac{1}{2} \to 0\}\}.$$

*This Space Intentionally Left Blank. Contributors include: Peter Doyle. 

This is assuming that we stop if I win the second toss. If we really play best-of-3, meaning we play out all 3 tosses no matter what, then the payoff is

$$\{\frac{1}{2} \to \{\frac{1}{2} \to a, \frac{1}{2} \to a\}, \frac{1}{2} \to \{\frac{1}{2} \to a, \frac{1}{2} \to 0\}\}.$$

These two situations are equivalent, and we want some transformation rules that will allow us to convert between the two representations.

## Transformation rules

The most basic transformation rules are those that allow us to permute comma-separated terms, e.g.

$$\{p_1 \to x_1, p_2 \to x_2\} \longleftrightarrow \{p_2 \to x_2, p_1 \to x_1\}.$$

We won't bother to spell out these rules formally.

In the following rules, $T, T_1, \ldots$ are metavariables representing probability trees.

Certainty:

$$\{1 \to T\} \longleftrightarrow T.$$

Pruning:

$$\{0 \to p_1, p_2 \to T_2, \ldots\} \longleftrightarrow \{p_2 \to T_2, \ldots\}.$$

Identical outcomes:

$$\{p_1 \to T, p_2 \to T, p_3 \to T_3, \ldots\} \longleftrightarrow \{(p_1 + p_2) \to T, p_3 \to T_3, \ldots\}$$

Conditioning:

$$p \to \{q_1 \to T_1, \ldots, q_n \to T_n\} \longleftrightarrow \{pq_1 \to T_1, \ldots, pq_n \to T_n\}.$$

These rules are sufficient to allow us to transform any gamble into *standard form*

$$\{p_1 \to x_1, \ldots, p_n \to x_n\},$$

where the outcomes $x_1, \ldots x_n$ are all distinct, and the probabilities $p_1, \ldots, p_n$ are all positive (and $\leq 1$, since their sum is necessarily 1).

## The expected value of a gamble

A *gamble* is a probability tree whose outcomes are all real numbers, representing payments. A gamble is *rational* if all branch probabilities are rational numbers. We call the gamble $x$, where $x \in \mathrm{R}$, a *constant gamble*, or an *ungamble*.

The *expected value* of a gamble is defined recursively as follows:

$$\mathrm{Val}(x) = x, \quad x \in \mathrm{R};$$

$$\mathrm{Val}(\{p_1 \to T_1, \ldots, p_n \to T_n\}) = p_1 \mathrm{Val}(T_1) + \ldots + p_n Val(T_n).$$

We wish to interpret $\mathrm{Val}(T)$ as the fair market value of the gamble $T$. To justify this, we are going to introduce one additional transformation rule, which changes a gamble only to the extent of incorporating a manifestly fair side bet.

Side bet:

$$\{p \to a, p \to b, p_3 \to T_3, \ldots, p_n \to T_n\} \longleftrightarrow \{p \to a{-}c, p \to b{+}c, p_3 \to T_3, \ldots, p_n \to T_n\}.$$

Like the previously introduced transformation rules, this rule preserves $\mathrm{Val}(T)$. With its help we may transform any rational gamble $T$ to (or from) an ungamble, which will necessarily be the constant gamble $\mathrm{Val}(T)$, the 'gamble' that always pays $\mathrm{Val}(T)$. Indeed, we can first transform $T$ to normal form, explode it into a normal form gamble where all probabilities are equal, use side bets to make all outcomes equal, and finally consolidate to an ungamble. We illustrate the procedure for Huygens's example of a gamble where you have two chance to win 8, and three chances to win 13.

$$\begin{aligned}
& \{\frac{2}{5} \to 8, \frac{3}{5} \to 13\} \\
\longleftrightarrow \quad & \{\frac{1}{5} \to 8, \frac{1}{5} \to 8, \frac{1}{5} \to 13, \frac{1}{5} \to 13, \frac{1}{5} \to 13\} \\
\longleftrightarrow \quad & \{\frac{1}{5} \to 8+8, \frac{1}{5} \to 0, \frac{1}{5} \to 13, \frac{1}{5} \to 13, \frac{1}{5} \to 13\} \\
\longleftrightarrow \quad & \{\frac{1}{5} \to 8+8+13, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 13, \frac{1}{5} \to 13\} \\
\longleftrightarrow \quad & \{\frac{1}{5} \to 8+8+13+13, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 13\}
\end{aligned}$$

$$
\begin{aligned}
&\longleftrightarrow \quad \{\frac{1}{5} \to 8+8+13+13+13, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0\} \\
&= \quad \{\frac{1}{5} \to 55, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0\} \\
&\longleftrightarrow \quad \{\frac{1}{5} \to 44, \frac{1}{5} \to 11, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0\} \\
&\longleftrightarrow \quad \{\frac{1}{5} \to 33, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 0, \frac{1}{5} \to 0\} \\
&\longleftrightarrow \quad \{\frac{1}{5} \to 22, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 0\} \\
&\longleftrightarrow \quad \{\frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 11\} \\
&\longleftrightarrow \quad \{1 \to 11\} \\
&\longleftrightarrow \quad 11
\end{aligned}
$$

This is not the most efficient method of reducing this gamble to an ungamble. What we did was to first convert the gamble to a *lottery*

$$\{1/n \to nc, 1/n \to 0, \ldots, 1/n \to 0\},$$

and then convert the lottery to an ungamble. This was done in honor of Huygens, who in addition to side bets allowed an additional transformation:

Lottery:

$$c \longleftrightarrow \{1/n \to nc, 1/n \to 0, \ldots, 1/n \to 0\}$$

In fact this transformation is redundant: As we have seen, it can be accomplished by means of side bets.

## Sources

This work is based on earlier works copyright 2003 Peter G. Doyle.