One ball to rule them all

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In the course of looking for new ways to solve the quintic using iterated rational maps, we are led to wonder, what is the Platonically ideal shape of a soccer ball?

The familiar 'association football ball' ('soccer ball', for short) is a round relative of a truncated icosahedron. Sanding the vertices off an icosahedron produces 12 regular pentagons where the vertices were, and converts the 20 triangular faces into hexagons. By sanding just the right amount, the hexagons can be made regular. But is this the ideal choice? A truncated icosahedron is not a Platonic solid, whose ideal shape is ordained by its symmetries. Perhaps we should truncate so as to minimize the disparity between the circumscribed and inscribed spheres, or maximize the volume of the associated ideal hyperbolic polyhedron?

As for the ideal soccer ball, should we simply inflate a truncated icosahedron, or should we look for an answer directly on the sphere?

We offer as a candidate the picture that emerges from iterating the rational map

$$z \mapsto \frac{(-z^{11} - 66z^6 + 11z)\, H_{20} - wzT_{30}}{(11z^{10} + 66z^5 - 1)\, H_{20} - wT_{30}},$$

where

$$H_{20} = -z^{20} + 228z^{15} - 494z^{10} - 228z^5 - 1,$$

$$T_{30} = z^{30} + 522z^{25} - 10005z^{20} - 10005z^{10} - 522z^5 + 1,$$

and the constant w is one of roots of the polynomial

$$512578125\, w^{12} - 3865218750\, w^{11} + 23152938750\, w^{10}$$
$$-74112921000\, w^{9} + 108824537925\, w^{8} - 149785512090\, w^{7}$$
$$+240751261832\, w^{6} - 293002530840\, w^{5} + 188253684000\, w^{4}$$
$$+132994820000\, w^{3} - 311513400000\, w^{2} + 139581600000\, w$$
$$-5632000000.$$