Trace Formula Lite: All you need is the Pythagorean theorem and the volume of a cylinder

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The Laplace spectrum via elementary geometry

The Selberg trace formula for a manifold M can be used to reduce questions about the Laplace spectrum to questions of elementary geometry. Thus, for example, showing that two manifolds are *isospectral* (have the same Laplace spectrum) can be as simple as computing the volume of a bunch of cylinders.

While this reduction of spectral theory to geometry is widely known, at least in principle, it seems to be widely ignored in practice. Here we will try to set the situation to rights, by developing the trace formula from scratch using only elementary geometry.

Well—elementary geometry and standard stuff about covering spaces and the like. But this extra stuff is only needed in developing the theory. In applying the theory, we will likely understand the universal coverings of the manifolds we're dealing with. This will allow us to dispense with any tools more sophisticated than the Pythagorean theorem and the volume of a cylinder. (Of course for hyperbolic manifolds, we'll need the hyperbolic Pythagorean theorem and the volume of a hyperbolic cylinder; but this is only to be expected.)

We remark that our whole discussion works just fine for orbifolds as well as for manifolds. But to keep the terminology palatable to readers who may not know or care much about orbifolds, we'll stick with manifolds.

One last thing: You will note that nowhere in this discussion of the trace formula do we actually write it down. That's because in practice, you can use the trace formula without knowing precisely what it is. In fact, it may be a distinct advantage not to know the formula. Perhaps the reason the formula has not been used more effectively is that potential beneficiaries have been mesmerized by the details of the formula, and missed the essential simplicity of the idea behind it.

The heat trace and the spectrum

Let M be a compact Riemannian manifold with Laplace spectrum $\lambda_0, \lambda_1, \ldots$ and corresponding orthonormal eigenfunctions $\phi_0, \phi_1, \ldots$.

The functions

$$u_n(x,t) = \phi_n(x) e^{-\lambda_n t}$$

satisfy the *heat equation*

$$\Delta u = \frac{\partial u}{\partial t}.$$

From these solutions we form the *heat kernel*

$$K(x,y;t) = \sum_n \phi_n(x)\phi_n(y) e^{-\lambda_n t}.$$

This kernel produces a solution to the heat equation from initial values given by an arbitrary function f on M:

$$u(x,t) = \int_M K(x,y;t) f(y) \mathrm{d}y = \sum_n < \phi_n, f > u_n(x,t),$$

with

$$u(x,0) = \sum_n < \phi_n, f > \phi_n(x) = f(x).$$

The *heat trace*

$$\mathrm{tr} K(t) = \int_M K(x,x;t) \mathrm{d}x$$

is determined by the Laplace spectrum:

$$\mathrm{tr}K(t) = \int_M K(x,x;t)\mathrm{d}x = \sum_n \int_M \phi(x)^2 \mathrm{d}x e^{-\lambda_n t} = \sum_n e^{-\lambda_n t}.$$

In fact, the heat trace is the Laplace transform of the *spectral measure* $\mathrm{d}\sigma$, which places a unit mass on the real line at the location of each eigenvalue λ_n:

$$\mathrm{tr}K(t) = \sum_n e^{-\lambda_n t} = \int_M e^{-st}\mathrm{d}\sigma(s),$$

where

$$\sigma(s) = |\{n : \lambda_n \leq s\}|\,.$$

By inverting the Laplace transform, we can recover the spectrum from the heat trace (at least in principle). Thus the heat trace and the spectrum determine one another.

The heat trace and the counting trace

Now assume that the universal cover $\bar{M}$ of M is homogeneous and isotropic, so that the heat kernel $\bar{K}(\bar{x},\bar{y};t)$ of $\bar{M}$ depends only on the distance $\mathrm{dist}(\bar{x},\bar{y})$ between $\bar{x}$ and $\bar{y}$:

$$\bar{K}(\bar{x},\bar{y};t) = k_t(\mathrm{dist}(\bar{x},\bar{y})).$$

For example, if M is a *platycosm* (compact flat 3-manifold) then $\bar{M} = \mathrm{R}^3$ and

$$k_t(r) = \frac{1}{(2\pi t)^{\frac{3}{2}}} e^{\frac{-r^2}{2t}}.$$

Using the time-honored *method of images*, we can push the heat kernel $\bar{K}$ on $\bar{M}$ down to M. Let Γ be the covering group, and let $\bar{x},\bar{y} \in \bar{M}$ be arbitrary lifts of $x, y \in M$.

$$K(x,y;t) = \sum_{\gamma\in\Gamma} \bar{K}(\bar{x},\gamma\bar{y};t) = \sum_{\gamma\in\Gamma} k_t(\mathrm{dist}(\bar{x},\gamma\bar{y})) = \int_0^\infty k_t(s)\mathrm{d}N_{x,y}(s),$$

where

$$N_{x,y}(s) = |\{\gamma \in \Gamma : \mathrm{dist}(\bar{x},\gamma\bar{y}) \leq s\}|\,.$$

We call $N(x,y;s) = N_{x,y}(s)$ the *counting kernel.* It counts the number of homotopically distinct paths from x to y of length $\leq s$. The corresponding measure $\mathrm{d}N_{x,y}(s)$ places a unit mass on the real line for each homotopy class of paths from x to y, located at the minimal length of curves in the given homotopy class.

Taking traces we have

$$\mathrm{tr}K(x, y) = \int_M K(x, x; t)\mathrm{d}x = \int_0^\infty k_t(s)\mathrm{d}\mathrm{tr}N(s),$$

where

$$\mathrm{tr}N(s) = \int_M N(x, x; s)\mathrm{d}x = \int_M |\{\gamma \in \Gamma : \mathrm{dist}(\bar{x}, \gamma\bar{x}) \le s\}|\,\mathrm{d}x.$$

We call $\mathrm{tr}N$ the *counting trace*. It counts how many points $x \in M$ which are 'within s of themselves'; this counting is done 'with multiplicity', so that homotopically distinct ways of getting from x back to x contribute independently to the count.

Evidently the counting trace determines the heat trace: Specifically, the heat trace is an integral transform of the associated counting measure $\mathrm{d}N(s)$. Previously, we saw that the heat trace was a transform of the spectral measure, and argued that we could recover the spectral measure from the heat trace by inverting the transform. Here again, we can invert the transform, so we can recover the counting trace from the heat trace. We won't worry here about the details of this inversion, because in practice, the important fact is that it can be done in principle.

We sum up this discussion as follows:

Theorem. On a manifold M whose universal cover $\bar{M}$ is homogeneous and isotropic, the Laplace spectrum, the heat trace, and the counting trace all determine one another: They capture exactly the same information about the manifold.

Corollary. Two compact manifolds with the same local geometry (assumed homogeneous and isotropic) and the same counting trace are isospectral.

Now we just need to figure out how to compute the counting trace.

Computing the counting trace

The counting trace measures how many points $x \in M$ are 'within s of themselves'. To compute it, we lump together ways of getting from x back to x according to the free homotopy class of the path involved.

$$\mathrm{tr}N(s) = \int_M |\{\gamma \in \Gamma : \mathrm{dist}(\bar{x}, \gamma\bar{x}) \le s\}|\, \mathrm{d}x = \sum_{[\gamma]\in[\Gamma]} \mathrm{Measure}(\{\hat{x} \in \bar{M}/Z(\gamma) : \mathrm{dist}(\hat{x}, \gamma\hat{x}) \le s\}).$$

Here $[\gamma]$ is the conjugacy class of γ; $[\Gamma]$ is the set of all conjugacy classes of Γ; and Measure is volume measure on the covering space $\bar{M}/Z(\gamma)$ of M corresponding to the subgroup $Z(\gamma)$ of all elements that commute with γ.

Because this last formula has a rather austere, group-theoretical cast to it, we hasten to explain how it works in particular cases. The situation is simplest when M is a space of negative curvature, for example the hyperbolic plane H^2 or hyperbolic space H^3. Then each free homotopy class on M contains a unique shortest representative, which is a closed geodesic, and the classes can be written