Integer-Point Enumeration in Polyhedra

Based on "Computing the Continuous Discretely" by Matthias Beck and Sinai Robins

What is a Polyhedron?

**Polyhedron**: A set, bounded or not, that can be described by the intersection of finitely many half-spaces and hyperplanes.

**Convex Polytope**: A convex hull of finitely many points in IR^d.

Equivalent to a bounded polyhedron

**Ex. Polyhedron in IR^2**

- Defined by hyperplanes:
  - \( L_1: \{ (x, y) \in \mathbb{R}^2 \mid y = -x + 1 \} \)
  - \( L_2: \{ (x, 0) \in \mathbb{R}^2 \} \)
  - \( L_3: \{ (0, y) \in \mathbb{R}^2 \} \)
  - \( P = \{ (x, y) \in \mathbb{R}^2 \mid y < x + 1, y \geq 0, x > 0 \} \)
If we include the lines $h_1, h_2, h_3$, then $\mathcal{P}$ becomes the convex polytope defined by

$$\mathcal{P} = \text{Conv} \mathcal{E} \{(0,0), (0,1), (1,0)\}$$

**What is the dimension of a Polytope?**

$$\dim \mathcal{P} = \dim(\text{Span } \mathcal{P} = \lambda x + \lambda' y) \| x, y \in \mathcal{P}, \lambda, \lambda' \in \mathbb{R} \}

The smallest space our polytope can comfortably reside in $\mathbb{R}^x$.

Fact: A $d$-dimensional polytope has at least $d+1$ vertices.

**Simplex:** a $d$-dimensional polytope with exactly $d+1$ vertices.
Triangulating Over a Polytope:

**Triangulation:** "Splitting up" a polytope into simplices of the same dimension

Fact: Triangulations are not necessarily unique

Thm: Every convex polytope can be triangulated using no new vertices

Why do we care about triangulation?

It makes all of our proofs easier!
**Integer-Point Enumeration**

\[ \mathbb{Z}^d = \{ (a_1, \ldots, a_d) \mid a_i \in \mathbb{Z} \} \text{ is the } d \text{-integer lattice.} \]

- A grid with a point whenever all coordinates are integers.

**Lattice-Point Enumeration \( L_P \):**

\[ L_P := \#(P \cap \mathbb{Z}^d) \]

- The number of integer points in \( P \).
- "Discrete volume."

**Dilation**

We can scale our polytope \( P \) by \( t \):

- \( tP \) is called the "\( t \)-th dilate" of \( P \).

\[ L_{tP}(t) = \#(tP \cap \mathbb{Z}^d) = \#(P \cap \mathbb{Z}^d) \]

**Examples**

- \( d = 1 \); \( P = [a, b] \), \( a, b \in \mathbb{Z} \)
- \( P = [2, 5] \)

\[ L_{[2,4]}(1) = 4 \]
\[ L_{[2,4]}(2) = 7 \neq 2(4) \]
Why study integer-point enumeration?

There is a deep connection between discrete and continuous volume
Discrete $\rightarrow$ Continuous

For a polytope $P \subset \mathbb{R}^d$:

- Draw $d$-dimensional boxes between the points of your integer lattice.
- $\#$ of boxes in $P \approx \text{Vol}(P)$
- Volume of each box $= 1^d = 1$

- Very rough volume approximation of $P = 1^d \cdot \text{Vol}(P)$

What if we dilate $P$ by scaling our lattice down?

- $\#$ of boxes in $P \approx \text{Vol}(P)$
- Volume of each box $= \frac{1}{t^d}$
- $\text{Vol}(P) \approx \frac{1}{t^d} \cdot (\text{Vol}(P))$

\[
\lim_{t \to 0} \frac{1}{t^d} \cdot (\text{Vol}(P)) = \text{Vol}(P)
\]

- What does this tell us about $O(\text{Vol}(P))$?

**Frobenius's Coin Problem**

Given $n$ coins of different values, what's the largest number we can't make using these coins?

- Coins: $a_1, \ldots, a_n \in \mathbb{Z}_{>0}$
- $P_1(n) = \# \{(m_1, \ldots, m_n) \in \mathbb{Z}^n_+ | m_1, \ldots, m_n \geq 0, a_1m_1 + a_2m_2 + \ldots + a_nm_n \leq n\}$
- Counts how many ways we can make $n$ with our coins
The Ehrhart Series

\[ \text{Ehr}_P(z) := 1 + \sum_{t \geq 1} h_P(t) z^t \]

**Ehrhart's Thm:**

If \( P \) is an integral convex \( d \)-polytope, then \( h_P(t) \) is a polynomial of degree \( d \).

**Proof Outline:**

**Lemma:**

If \( \sum_{t \geq 0} f(t) z^t = \frac{g(z)}{(1-z)^{d+1}} \)

then \( f \) is a polynomial of degree \( d \)

\[ \iff \]

\( g \) is a polynomial of degree \( d \), \( g(1) \neq 0 \)

Coning over a Polytope

\[ \text{Integral Point Transform: } \nu_S(z) = \nu_S(z_1, z_2, \ldots, z_{d}) = \sum_{\nu \in \mathbb{Z}^m} z^\nu \]
Calculating $\text{Ehr}_\phi(z)$

If $\text{Ehr}_\phi(z) = 1 + \sum \frac{f(\text{tl}) z^e}{(1-z)^{d+1}} = g(z)$

then our Ehrhart series is uniquely determined by $g(z)$, a polynomial of degree $d$.

For integral convex $d$-polytopes:

Let $g(z) = h_d z^d + h_{d-1} z^{d-1} + \ldots + h_1 z + h_0$

$\Rightarrow h_0 = 1$

$\Rightarrow h_1 = h_\phi(1) - d - 1$

Similarly, there are formulas for $h_2, h^*, \ldots$

based on $h_\phi(z), h_\phi(3), \ldots$

So we can calculate $g(z)$, and therefore $\text{Ehr}_\phi(z)$ by looking at the first $d$ dilates of $\phi$. 
We can actually do better!

**Theorem:**

If \( P \) is an integral convex \( d \)-polytope with

\[
\text{Ehr}_P(z) = h^*_d z^d + h^*_{d-1} z^{d-1} + \cdots + h^*_1 z + 1
\]

Then \( d = h^*_1 = \cdots = h^*_n = 0 \)

\[
\iff
\]

\((d-k+1)P\) is the smallest integer dilate of \( P \) that contains an interior lattice point.

**Proof Outline**

**Ehrhart-Macdonald Reciprocity:**

Suppose \( P \) is a convex rational polytope

\[
\Rightarrow h_P(-t) = (-1)^{d+1} h^*_P(t)
\]

Where \( P^* \) is the interior of \( P \).
Big Takeaway:

To calculate the Ehrhart Series for a polytope $P$, we only need to look at the dilates of $P$ until we find an integer lattice point in the interior (or at worst $d$ of them).

Remember:

Since $Ehr_P(z)$ is encoded by $h^P(t)$, we can find $h^P(t)$ for any dilate $t$ by looking at the coefficient of $z^t$ in the Ehrhart Series.
\[ Eh_{\phi}(z) = \frac{g(z)}{1-z^3} = \frac{h^*_x z^2 + h^*_1 z + 1}{(1-z)^3} \]

\[ h_1 = h(1) - d - 1 = 4 - 2 - 1 = 1 \]

\[ d - k + 1 = 2 \]

\[ 2 - k + 1 = 2 \]

\[ k = 1 \]
What about Rational Polytopes?

**Quasi polynomials**: Periodic functions that alternate between polynomials (constituents).

**Degree** of a quasi polynomial $Q(t)$ is the highest degree among its constituents.
Ehrhart's Theorem for Rational Polytopes

If $P$ is a rational convex $d$-polytope

Then $L_P(t)$ is a quasi-polynomial in $t$ of degree $d$.

The period of $Q$ divides the lcm of the denominators of the coordinates of the vertices of $P$.

**Ex.**

<table>
<thead>
<tr>
<th>$t$</th>
<th>$L_P(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>

$L_P(t) = \begin{cases} \frac{(t+1)(t+1)}{2} & : t \text{ odd} \\ \frac{(t+1)(t+2)}{2} & : t \text{ even} \end{cases}$