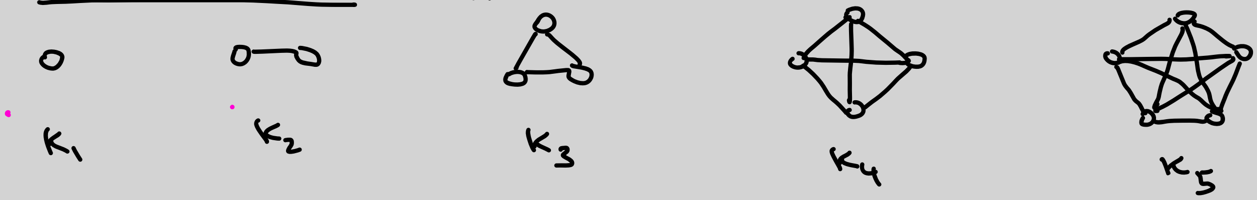
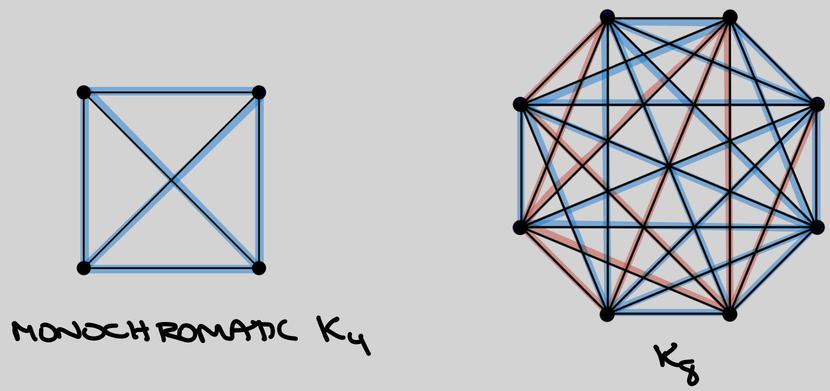
THE PROBABILISTIC METHOD

MOTIVATION:

- THE COMPLETE GRAPH K_n



- 2-EDGE COLORING OF K_n



? HOW MANY COPIES OF A MONOCHROMATIC K_4 ARE IN THE 2-EDGE COLORING OF K_8 ABOVE?

? AS n GETS LARGE, NUMBER OF COPIES CAN BE LARGE FOR AN ARBITRARY EDGE COLORING OF K_n... CAN WE FIND AN EDGE-COLORING OF K_n SUCH THAT THE NUMBER OF COPIES IS BOUNDED?

PROPOSITION:

FOR EVERY INTEGER n, THERE EXISTS A COLORING OF THE EDGES OF THE COMPLETE GRAPH K_n BY TWO COLORS SO THAT THE TOTAL NUMBER OF MONOCHROMATIC COPIES OF K_4 IS AT MOST $\binom{n}{4} \cdot 2^{-5}$.

BETTER BE $< \binom{n}{4}$

PROOF: ??

THE METHOD:

GOAL: PROVE THAT A STRUCTURE WITH CERTAIN DESIRED PROPERTIES EXISTS

HOW:

1. DEFINE AN APPROPRIATE PROBABILITY SPACE OF STRUCTURES
2. SHOW THAT THE DESIRED PROPERTIES HOLD IN THESE STRUCTURES WITH POSITIVE PROBABILITY

PROOF OF EARLIER PROPOSITION: (PROB. METHOD)

- CONSIDER A RANDOM TWO-COLORING OF THE EDGES OF K_n OBTAINED BY COLORING EACH EDGE INDEPENDENTLY RED OR BLUE, WHERE EACH COLOR IS EQUALLY LIKELY.
- FOR ANY FIXED SET R OF 4 VERTICES, LET A_R BE THE EVENT THAT THE INDUCED SUBGRAPH OF K_n ON R IS MONOCHROMATIC.
- FOR FIXED K_n, $Pr[A_R] = 2^{-5}$ AND THERE ARE $\binom{n}{4}$ POSSIBLE CHOICES FOR R... THE EXPECTED NUMBER OF MONOCHROMATIC COPIES OF K_4 IS $\binom{n}{4} 2^{-5}$.
- IF THE "AVERAGE" NUMBER OF COPIES IS $\binom{n}{4} 2^{-5}$, THEN CERTAINLY THERE EXIST SOME COLORING WITH AT MOST THAT MANY COPIES! $\square$

COMMENTS

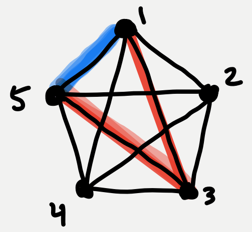
1. UPPER BOUND IS MEH
2. WHICH COLORING(S) OF K_n HAVE THIS PROPERTY?!?

DERANDOMIZATION (TACKLING ②)

? CAN WE ACTUALLY FIND DETERMINISTICALLY SUCH A COLORING IN TIME WHICH IS POLYNOMIAL IN n?

YES... GREEDY APPROACH: METHOD OF CONDITIONAL PROBABILITIES

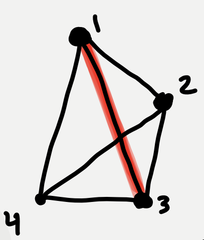
EX:



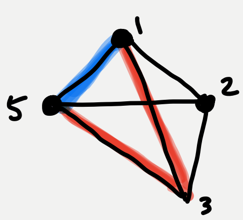
K_5

WEIGHT FUNCTION w ASSIGNS NUMBER TO PARTIALLY COLORED K_n AS FOLLOWS:

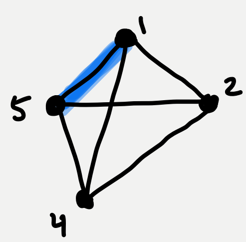
- FOR EACH COPY K OF K_4 IN K_n ($\leftarrow 5$) DEFINE $w(K)$ AS FOLLOWS:
  - IF ≥ 1 EDGE OF K IS COLORED RED AND AT LEAST ONE EDGE OF K IS COLORED BLUE, THEN $w(K) = \underline{0}$.
  - IF NO EDGE OF K IS COLORED, THEN $w(K) = \underline{2^{-5}}$.
  - IF $r \geq 1$ EDGES OF K ARE COLORED, ALL WITH THE SAME COLOR, THEN $w(K) = \underline{2^{r-6}}$.
- $$W = \sum_{\substack{K = K_4 \\ K_4 \subseteq K_n}} w(K)$$



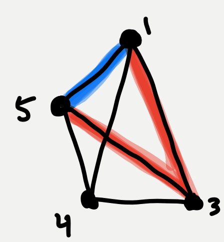
$w(K) = 2^{-5}$



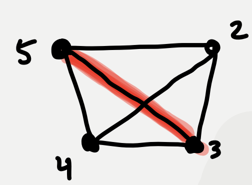
$w(K) = 0$



$w(K) = 2^{-5}$



$w(K) = 0$



$w(K) = 2^{-5}$

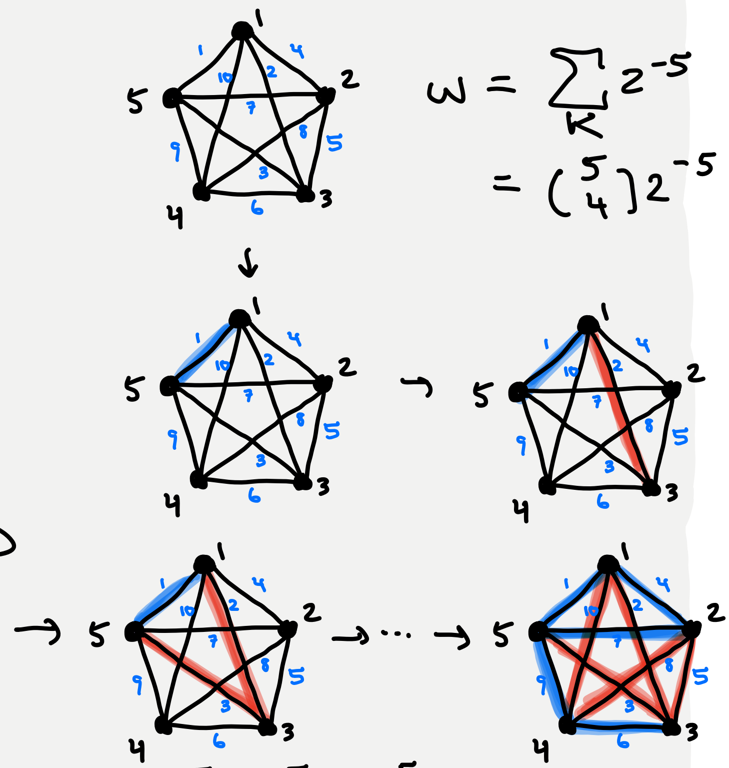
$$W = 3 \cdot 2^{-5} = 3/32$$

ANALOG: $w(K) = \Pr[K \text{ WILL BE MONOCHROMATIC}]$

W = EXPECTED NUMBER OF MONOCHROMATIC COPIES OF K_4 WHEN COLORING IS COMPLETE

WE NOW DESCRIBE A DETERMINISTIC ALGORITHM, WHICH, FOR ANY n, FINDS A 2-COLORING OF K_n SUCH THAT THE NUMBER OF MONOCHROMATIC COPIES OF K_4 IN K_n IS AT MOST $\binom{n}{4} \cdot 2^{-5}$.

- ORDER THE $\binom{n}{2}$ EDGES OF K_n ARBITRARILY.
- WHILE $e_{\binom{n}{2}}$ HAS NOT BEEN COLORED
  - SUPPOSE $e_1, \ldots, e_{i-1}$ ARE ALREADY COLORED AND e_i HAS YET TO BE COLORED
  - LET w BE THE WEIGHT OF K_n w.r.t. COLORING SO FAR
  - COMPUTE w_{Red}, w_{Blue}
  - COLOR e_i $\min(w_{Red}, w_{Blue})$



$$w = \sum_K 2^{-5} = \binom{5}{4} 2^{-5}$$

$$w' = 3 \cdot 2^{-5} < \binom{5}{4} 2^{-5}$$

$$w = 0$$

At end, $w = \#$ MONO. COPIES

NOTE: $w = \frac{1}{2} w_{Red} + \frac{1}{2} w_{Blue}$

OBS: w NEVER INCREASES SINCE $w = \frac{w_{Red} + w_{Blue}}{2}$

* TRIVIAL RUNNING TIME: $\binom{n}{2} \cdot \binom{n}{4} \Rightarrow O(n^6)$

NOTE: $2^{\binom{n}{2}} \in O(2^{n^2})$ 2-EDGE COLORINGS OF K_n $\Rightarrow$ TRIVIAL ENUMERATION APPROACH IS NOT POLYNOMIAL IN n.

REFERENCES

The Probabilistic Method (Alon, Spencer)

ACKNOWLEDGEMENTS

Mentor: Kathy Lin (Thank you!)