A taste of Intuitionistic Logic

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To \( \varphi \) or not to \( \varphi \)

**Law of excluded middle**

\[ \varphi \lor \neg \varphi \]
Natural Deduction

Let $\mathbf{PV}$ be an infinite set of *propositional variables*.

**Definition**

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
<tr>
<td>Let $\Delta$ be the least set such that:</td>
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<tr>
<td>- $\bot \in \Delta$</td>
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<tr>
<td>- $\mathbf{PV} \subseteq \Delta$</td>
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<tr>
<td>- $\phi, \psi \in \Delta$ then $\phi \land \psi, (\phi \lor \psi), (\phi \rightarrow \psi) \in \Delta$</td>
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$\Delta$ is our set of formulas.
A **judgment** is a pair consisting of a finite set of formulas $\Gamma$ and a formula $\varphi$, and we denote it by $\Gamma \vdash \varphi$. 

Definition
Natural Deduction

Classical Propositional Calculus

\[
\begin{align*}
\Gamma, \varphi &\vdash \varphi \quad \text{(Ax)} & \Gamma &\vdash \varphi \lor \neg \varphi \quad \text{(Ax)} \\
\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} &\quad \text{($\to$I)} & \frac{\Gamma \vdash \varphi \to \psi, \Gamma \vdash \varphi}{\Gamma \vdash \psi} &\quad \text{($\to$E)} \\
\frac{\Gamma \vdash \varphi, \Gamma \vdash \psi}{\Gamma \vdash \varphi \land \psi} &\quad \text{($\land$I)} & \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi} &\quad \text{($\land$E)} \\
\frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi} &\quad \text{(VI)} & \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \lor \psi} &\quad \text{(VE)} \\
\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} &\quad \text{(⊥E)}
\end{align*}
\]
Natural Deduction

**Definition**

We inductively define a **derivable judgment** as any judgment that is either an axiom or is derived from the rules of inference.

**Definition**

A **theorem** is a derivable judgment with $\Gamma = \emptyset$. 
Natural Deduction
Natural Deduction

Classical Propositional Calculus

\[
\begin{align*}
\Gamma, \varphi & \vdash \varphi (Ax) & \Gamma & \vdash \varphi \lor \neg \varphi (Ax) \\
\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} & (\rightarrow I) & \frac{\Gamma \vdash \varphi \rightarrow \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} & (\rightarrow E) \\
\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \land \psi} & (\land I) & \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi} (\land E) & \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \psi} \\
\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \lor \psi} & (\lor I) & \frac{\Gamma, \varphi \vdash \theta \quad \Gamma, \psi \vdash \theta}{\Gamma \vdash \varphi \lor \psi} & (\lor E) \\
\frac{\Gamma \vdash \bot}{\Gamma \vdash \psi} & (\bot E) & \frac{\Gamma \vdash \varphi}{\Gamma \vdash \theta} & (\bot E)
\end{align*}
\]
Natural Deduction

*Intuitionistic Propositional Calculus*

\[
\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \quad (\rightarrow I)
\]

\[
\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \land \psi} \quad (\land I)
\]

\[
\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \lor \psi} \quad (\lor I)
\]

\[
\frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \lor \psi} \quad (\lor E)
\]

\[
\frac{\Gamma \vdash \psi}{\Gamma \vdash \psi} \quad (\land E)
\]

\[
\frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi} \quad (\land E)
\]

\[
\frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \lor \psi} \quad (\lor E)
\]

\[
\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \quad (\bot E)
\]
Natural deduction

Intuitionistic Propositional Calculus
From Classical to Intuitionistic

What is the difference?
Semantics of Classical Propositional Calculus

\( \neg(p \land q) \rightarrow \neg \neg p \lor \neg q \)
Semantics of Classical Propositional Calculus

Definition

A classical valuation is a function from PV to \( \{0, 1\} \).

Definition

Given a valuation \( v \), we define the value function \( V : \Delta \rightarrow \{0, 1\} \) as:

- \( V(\bot) = 0 \)
- \( V(\phi) = v(\phi) \) if \( \phi \in PV \)
- \( V(\phi \land \psi) = \min\{V(\phi), V(\psi)\} \)
- \( V(\phi \lor \psi) = \max\{V(\phi), V(\psi)\} \)
- \( V(\phi \rightarrow \psi) = \mathbb{1}_{V(\phi) \leq V(\psi)} \)
Semantics of Classical Propositional Calculus

Definition

We say a formula $\phi$ is classically valid and write it as $\vdash \phi$ whenever for every valuation $v$ we have $V(\phi) = 1$. 
A partial order \( \{ H, \leq \} \) is a **Heyting algebra** if:

- Every two elements \( a, b \in H \) have a supremum \( (a \cup b) \) and an infimum \( (a \cap b) \) in \( H \).
- Every two elements \( a, b \in H \) have a relative pseudo complement \( (a \rightarrow b) \), which is the greatest \( c \in H \) such that \( a \cap c \leq b \).
- \( H \) has both top (1) and bottom (0) elements.
Semantics
Semantics

Definition

Given a Heyting algebra $\mathcal{H} = \{ H, \leq, \cup, \cap, 0, 1, \Rightarrow \}$, an intuitionistic valuation is a function from $\mathbf{PV}$ to $H$.

Definition

Given a Heyting algebra $\mathcal{H} = \{ H, \leq, \cup, \cap, 0, 1, \Rightarrow \}$ and a valuation $v$, we define the value function $V : \Delta \rightarrow H$ as:

- $V(\bot) = 0$
- $V(\varphi) = v(\varphi)$ if $\varphi \in \mathbf{PV}$
- $V(\varphi \land \psi) = V(\varphi) \cap V(\psi)$
- $V(\varphi \lor \psi) = V(\varphi) \cup V(\psi)$
- $V(\varphi \rightarrow \psi) = V(\varphi) \Rightarrow V(\psi)$
Semantics

Definition

We write $\models \varphi$ whenever we have that $V(\varphi) = 1$, for every Heyting algebra $\mathcal{H}$ and every valuation $v$. 
Semantics

Theorem

$\vdash \varphi$ if and only if $\models \varphi$. 
Semantics

Non-redundancy of the Law of Excluded Middle

Theorem

\( \neg p \lor \neg \neg p \)

Proof.
Semantics

Theorem

\[ \forall \neg
\neg p \rightarrow p \]

Proof.
Glivenko’s Theorem

Theorem

A formula $\varphi$ is classically valid if and only if $\neg\neg\varphi$ is intuitionistically valid.
That’s all

Questions?