DIAGRAMS MOD p AND INTEGRAL STRUCTURES IN REPRESENTATIONS OF REDUCTIVE GROUPS OF SEMISIMPLE RANK ONE

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Abstract. Let p be a prime, and let F be a finite extension of $\mathbb{Q}_p$. The local Langlands correspondence associates certain packets of representations of (the F-points of) a reductive group G with certain conjugacy classes of homomorphisms $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to {}^{\mathrm{L}}G$. When $G = GL_2(\mathbb{Q}_p)$, such a correspondence was built between representations on p-adic Banach spaces and 2-dimensional Galois representations. For other groups (e.g. $GL_2(F)$ where $F \neq \mathbb{Q}_p$, or $GL_n(F)$ for $n > 2$) such a correspondence has not been found. One of the main tools in establishing the correspondence for $GL_2(\mathbb{Q}_p)$ was the existence of integral structures in certain locally algebraic representations of $GL_2(\mathbb{Q}_p)$. We prove criteria for the existence of such norms in certain locally algebraic representations of groups of semisimple rank one, defined over F. This both gives simpler proofs of the author's previous results (for $U_3(F)$ and $GL_2(F)$) and generalizes them. This builds on Hu's work for $GL_2(F)$ and generalizes it. We will also present some computational aspects involved in this research and future research.

1. The Problem

1.1. **Self introduction.** Self introduction and thank the organizers. work in progress.

1.2. **The setup and the problem.** Let p be a prime number and F a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_F$ be the ring of integers.

Let $\mathbb{G}$ be a reductive group over F, $G = \mathbb{G}(F)$ its group of F-points. Let E/F be a finite extension such that G splits over E.

Let C be a finite extension of $\mathbb{Q}_p$ such that $|Hom(E,C)| = [E : \mathbb{Q}_p]$, and Π a locally algebraic representation of G with coefficients in C. (Note: C is p-adic!!!)

Definition 1.1. Let Π be a representation of G over C. An *integral structure* in Π is an $\mathcal{O}_C$ -submodule of Π, stable under G, which spans Π over C, and contains no C-line.

Example 1.2. Let $G = GL_2(F)$, and let $\Pi = \{f : \mathbb{P}^1(F) \to C\}$ with action by right translation, induced from the action of G on $\mathbb{P}^1$ - $(gf)(x) = f(xg)$. Then $\Lambda = \{f : \mathbb{P}^1(F) \to \mathcal{O}_C\}$ is an integral structure.

This is equivalent to asking whether Π admits a nonzero p-adic unitary completion, by the following argument.

Lemma 1.3. Π *admits an integral structure if and only if* Π *admits a nonzero p-adic G-equivariant unitary completion.*

Proof. If Π admits a nonzero p-adic unitary completion, let Π_0 be the unit ball - $\Lambda = \{x \in \Pi \mid |x| \le 1\}$, where $|\cdot|$ is the G-equivariant norm on the completion. It is an integral structure.

Conversely, if Π admits an integral structure Λ, consider its *gauge*, $\lambda : \Pi \to \mathbb{R}^{\ge 0}$, defined by $\lambda(x) = q_C^{-v_\Lambda(x)}$, where $v_\lambda(x) = \sup\{v \mid x \in \varpi_C^v \Lambda\}$. This is a seminorm, and since Λ contains no C-line, a norm.

Since Λ is G-equivariant, it is also G-equivariant, and completing Π with respect to this G-equivariant norm, we obtain a nonzero p-adic G-equivariant unitary completion. $\square$

There is also a natural notion of equivalence between integral structures.

Definition 1.4. Two integral structures $\Lambda_1, \Lambda_2 \subseteq \Pi$ are *commensurable* if there exists constants $\alpha, \beta \in C^\times$ such that $\alpha\Lambda_1 \subseteq \Lambda_2 \subseteq \beta\Lambda_1$.

It is also not difficult to see that:

Lemma 1.5. *The integral structures Λ_1, Λ_2 in Π are commensurable if and only if they induce the same unitary completion of* Π.

An interesting question is then:

Problem 1.6. What are the commensurability classes of integral structures in Π?

One may even restrict to the easier question:

Problem 1.7. Is there an integral structure in Π?

In general, it is a central and difficult open problem to decide whether there exist integral structures in Π.
The first non-trivial examples were found by Breuil [2] in the case of $GL_2(\mathbb{Q}_p)$.
An obvious necessary condition for the existence of integral structures is that the central character of Π is unitary. (Else you have a C-line in any submodule by the action of the center).
Emerton's theory of Jacquet functors on locally analytic representations provides other necessary conditions, and conjecturally, these conditions with the unitarity of the central character are also sufficient.
This is related to the Breuil-Schneider conjecture, which turns out to be very difficult to prove in general.

2. Motivation

2.1. **The Langlands Programme.** Let me try to give a (very) brief overview of the Langlands programme. Our focus will remain local in the rest of the talk.

2.1.1. *The modularity theorem.* Recall that the modularity theorem states that every elliptic curve E, defined over $\mathbb{Q}$, is modular. Namely, if E is of conductor N, then there exists a modular form f of weight 2 and level N, such that $L(E,s) = L(f,s)$.
This has many far reaching consequences, since $L(f,s)$ has analytic continuation, establishing Artin's conjecture for $L(E,s)$ and other well known conjectures (now theorems).
This equality of L functions is not merely numerical, but comes from a deeper phenomenon -
Let l be a prime, and consider $H^1_{et}(E, \mathbb{Q}_l)$ (can think also on the Tate module $T_l(E) = \lim_{n\to\infty} E[l^n] \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$). It is a vector space over $\mathbb{Q}_l$ which admits a natural action of the Galois group $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.
When l is a prime of good reduction, this is a 2-dimensional vector space, and we obtain a Galois representation $\rho_E : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_l)$, and $L(\rho_E, s) = L(E,s)$, and even

$$L_p(\rho_E, s) = L(\rho_E \mid_{G_{\mathbb{Q}_p}}, s) = L_p(E,s)$$

explaining the local Euler factors of $L(E,s)$ as coming from representations of the decomposition groups.
On the other side, a modular form $f : \mathcal{H} \to \overline{\mathbb{Q}}_l$ of weight k and level N gives rise to a map $\varphi_f : GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) \to \overline{\mathbb{Q}}_l$ via the strong approximation theorem:

$$\varphi_f(\gamma h_\infty k) = \det(h_\infty)^{k/2}(ci+d)^{-k} f(h_\infty \cdot i)$$

Here $\gamma \in GL_2(\mathbb{Q})$, $h_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $k \in K_0(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \mid C \equiv 0 \mod N \right\}$.
This function φ_f is in fact, automorphic, and when f is cuspidal, φ_f is bounded, hence belongs to the Hilbert space $L^2(Z(\mathbb{A}) \cdot GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}), \overline{\mathbb{Q}}_l)$. Denote the closed subspace generated by φ_f by π_f.
Then π_f is an automorphic representation of $GL_2(\mathbb{A})$, and by the tensor product theorem can be written as $\pi_f = \hat{\bigotimes}_p \pi_{f,p}$ where $\pi_{f,p}$ is a smooth irreducible admissible representation of $GL_2(\mathbb{Q}_p)$.
One can then construct an L-series for these representations, and when f is an eigenform, we have

$$L_p(\pi_f, s) = L(\pi_{f,p}, s) = L_p(f,s)$$

explaining the Euler factors as coming from representation of $GL_2(\mathbb{Q}_p)$.
In this specific case, of elliptic curves over $\mathbb{Q}$, we may even note that by looking at the cohomology of the modular curve $H^1_{et}(X_0(N), \overline{\mathbb{Q}}_l)$, we have a Hecke action, commuting with the Galois and the $GL_2(\mathbb{A})$ action.
The Hecke algebra is further commutative, hence its irreducible representations are one dimensional, and one may decompose the space $H^1_{et}(X_0(N), \overline{\mathbb{Q}}_l)$ according to these characters λ (systems of Hecke eigenvalues).
One then may decompose these isotypic subspaces $H^1_{et}(X_0(N), \overline{\mathbb{Q}}_l)[\lambda]$ as representations of $G_{\mathbb{Q}} \times GL_2(\mathbb{A})$.
We will obtain that the irreducible components are of the form $\rho_E \otimes \pi_f$, giving rise to a correspondence between the representations ρ_E, i.e. 2-dimensional representations of the Galois group $G_{\mathbb{Q}}$ over $\overline{\mathbb{Q}}_l$, and the representations π_f,
irreducible automorphic representations of $GL_2(\mathbb{A})$.

2.1.2. *The Langlands conjectures.* The Langlands programme predicts that there exists such a correspondence in a much more general setting.

That is, if G is a reductive group, defined over a global field K, then one expects to find a correspondence between certain (equivalence classes of) homomorphisms $\rho : G_K \to {}^L G(\overline{\mathbb{Q}}_l)$ on the one side ("Galois representations"), and certain irreducible automorphic representations $\pi : G(\mathbb{A}_K) \to H$ on the other side ("reductive representations"). Moreover, this correspondence is expected to be functorial in a certain sense.

This conjecture is still open even for $GL_2/\mathbb{Q}$.

It is a theorem for G defined over a function field (Lafforgue).

For GL_n over a field which is either totally real or CM, and automorphic representations which are regular at infinity, we now know to associate Galois representations.

The next frontier seems to be able to deal with non-regular algebraic automorphic representations, the simplest case of which being algebraic Maass forms for GL_2 .

Where do we stand for $GL_2/\mathbb{Q}$?

We try to find a correspondence between

(1) the set of algebraic at infinity cuspidal automorphic representations for $GL_2(\mathbb{A})$.

(2) the set of 2-dimensional continuous irreducible Galois representations of $G_{\mathbb{Q}}$ over $\overline{\mathbb{Q}}_l$ which are unramified at almost all primes and **de-Rham at l.**

We will conme back to this last condition.

One can divide (1) to three classes by the type of π_∞. (1a) - it is a discrete series, (1b) - it is the limit of discrete series representations, (1c) it is a principal series.

For (a),(b) we have maps (1) $\to$ (2) constructed by Deligne and Deligne-Serre. One would like to find a similar map for (c) (the case of Maass forms) and prove surjectivity. (injectivity will follow from multiplicity one).

The maps (a),(b) should have as image all odd Galois representations. The image of (1a) surjects on the set (2a) of representations with distinct Hodge-Tate weights (This is the part of Fontaine-Mazur conjecture proved by Kisin and Emerton).

We don't know yet the surjectivity of (1b) to (2b) but special cases have been doen by Buzzard, Gee, Taylor and Calegari.

2.1.3. *The Local Langlands correspondence.* The conjectured correspondence between Galois representations and reductive representations should have a good notion of local-global compatiblity. That is, the local factors should correspond (not only the L_p, but in fact even the ϵ and Γ factors).

Namely, one expects that the representations π_p of the reductive group at the place p, will correspond to the representations of the Decomposition group at p, $\rho\mid_{G_p}$.

For GL_n over a local field, this is a theorem. The archimedean cases were done by Langlands and Tunnels in the 70's, and the function field was done (even globally) by Lafforgue. The most difficult case was of a p-adic field, and it remaind open until 2001.

Before we state it, we will replace the notion of Galois representations by this of a Weil-Deligne representation.

Definition 2.1. A *Frobenius semi-simple Weil-Deligne Representation of the Weil group, W_F of F, is a pair (r, N) where r is a semi-simple representation of W_F on a finite dimenional vector space V, which is trivial on an open subgroup, and $N \in End(V)$ is such that $r(\sigma)Nr(\sigma)^{-1} = \left|Art_F^{-1}(\sigma)\right|_F \cdot N$ for all $\sigma \in W_F$, where $Art_F : F^\times \to W_F^{ab}$ is Artin's reciprocity law from local class field theory, normalized so that geometric Frobenius elements map to uniformizers.*

Note that any Galois representation gives rise to such a representation by Grothendieck's monodromy theorem. (structure of Galois group + trivial action of wild inertia), and this data suffices to construct L-functions.

!!! Maybe don't write everything down, just that it satisfies nice properties.

Theorem 2.2. *(Harris, Taylor 2001, Henniart 2001) Let F be a finite extension of $\mathbb{Q}_p$. Let $Irr(GL_n(F))$ denote the set of isomorphism classes of irreducible smooth representations of $GL_n(K)$ over $\mathbb{C}$ $(\overline{\mathbb{Q}}_l, l \neq p)$. Let $WDRep_n(W_F)$ denote the set of isomorphism classes of n-dimensional Frobenius semi-simple Weil-Deligne representations of the Weil group W_F, of F over $\mathbb{C}$ $(\overline{\mathbb{Q}}_l$,$l \neq p)$. There exists a correspondence $rec_F : Irr(GL_n(F)) \to WDRep_n(W_F)$ such that:*

(1) If $\pi \in Irr(GL_1(F))$, then $rec_F(\pi) = \pi \circ Art_F^{-1}$. (It extends local class field theory).

(2) If $[\pi_i] \in Irr(GL_{n_i}(F))$, then $L(\pi_1 \times \pi_2, s) = L(rec_F(\pi_1) \otimes rec_F(\pi_2), s)$. (and the same for ϵ-factors). (compatible with products).
(3) If $[\pi] \in Irr(GL_n(F))$ and $\chi \in Irr(GL_1(F))$, then $rec_F(\pi \otimes (\chi \circ \det)) = rec_F(\pi) \otimes rec_F(\chi)$. (a particular case of functoriality)
(4) If $[\pi] \in Irr(GL_n(F))$ and π has central character χ, then $\det rec_F(\pi) = rec_F(\chi)$.
(5) If $[\pi] \in Irr(GL_n(F))$, then $rec_F(\pi^\vee) = rec_F(\pi)^\vee$. (compatible with contragredient)

However, this was still not enough to do something for the global conjecture. Even for the modularity theorem, one needs to understand the behaviour at all(!!!) primes. (e.g. the famous paper wild 3-adic exercises).

This leads to our next topic.

2.1.4. *The p-adic Local Langlands programme.* The original aim of the p-adic local Langlands programme was to look for a possible p-adic analogue of the classical (and l-adic local correspondence).

The p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ was fully developed, essentially by Berger, Breuil and Colmez, using the theory of (φ, Γ)-modules. It was only fully completed in 2014. after further work of Paskunas and Dospinescu on small primes.

For some "nice" ("potentially semistable") Galois representations $\rho : G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}}_p)$, it is possible to attach a smooth representation $\pi_{sm}(\rho)$ of $GL_2(\mathbb{Q}_p)$ much like as in the classical setting. However, the map $\rho \rightsquigarrow \pi_{sm}(\rho)$ is no longer reversible.

The reason is that there are many more Galois representations in characteristic p, since the wild inertia may act non-trivially, e.g. the p-adic cyclotomic character.

This suggested that one should enlarge the category of representations on the reductive ("automorphic") side.

One attempt is this - the Hodge-Tate weights (basically the powers of the cyclotomic character appearing in the representation) correspond to dominant algebraic weights of $GL_2(\mathbb{Q}_p)$, and thus we may construct an irreducible algebraic representaiton $\pi_{alg}(\rho)$ associated to ρ, defined over $\mathbb{Q}_p$.

Still, one cannot reconstruct ρ from $\pi_{sm}(\rho)$ and $\pi_{alg}(\rho)$.

The problem is that although the Galois representation ρ, can be classified purely in terms of some linear algebraic data. That data includes also a filtration, called the Hodge filtration, which is lost when constructing $\pi_{sm}(\rho)$ and $\pi_{alg}(\rho)$.

Note that as the coefficient field is p-adic, the two representations "live" in the same universe, and we may tensor them to form $\pi_{sm}(\rho) \otimes \pi_{alg}(\rho)$.

This representation is no longer smooth nor is it algebraic, but it is "locally algebraic" - every vector has an open neighbourhood in which G acts on it polynomially.

The p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ takes any continuous representation $\rho : G_{\mathbb{Q}_p} \to GL_2(C)$, and attaches to it a Banach C-space $\Pi(\rho)$ with a unitary $GL_2(\mathbb{Q}_p)$-action. The map $\rho \rightsquigarrow \Pi(\rho)$ is reversible, and compatible with classical local Langlands in the following sense. If ρ is potentially semistable, its subspace of locally algberaic vectors is

$$\Pi(\rho)^{alg} = \pi_{alg}(\rho) \otimes \pi_{sm}(\rho)$$

Furthermore, otherwise $\Pi(\rho)^{alg} = 0$.

Moreover, when ρ is irreducible, $\Pi(\rho)^{alg}$ is dense in $\Pi(\rho)$, so that $\Pi(\rho)$ can be obtained from $\pi_{sm}(\rho) \otimes \pi_{alg}(\rho)$ by completing with respect to the norm.

The different completions of $\pi_{sm}(\rho) \otimes \pi_{alg}(\rho)$ to such Banach spaces, are in bijection with the possible Hodge filtrations on Galois representations with these Hodge-Tate weights and Weil-Deligne representation.

3. The Breuil Schneider conjecture

For groups other than $GL_2(\mathbb{Q}_p)$ very little is known. One of the main conjectures was stated in ([?]) by Breuil and Schneider, and in some sense it is a "first approximation" - for certain $\rho : Gal(\overline{\mathbb{Q}}_p/F) \to GL_n(C)$, one can define the representation $BS(\rho) := \pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$, and if it is indeed a subrepresentation of some larger, conjectural, Banach C-space $\Pi(\rho)$ with a unitary $GL_n(F)$-invariant norm, it should admit an invariant $GL_n(F)$-invariant norm. The resulting completions should be closely related to the yet undefined $\Pi(\rho)$ - at least in the irreducible cases.

We say that a representation V is locally algebraic if $V = V^{alg}$.

Conjecture 3.1. *(Breuil, Schneider [?]) The representation V arises from a potentially semistable Galois representation if and only if V admits a $GL_n(F)$-invariant norm.*

The "if" part is completely known for $GL_n(F)$ ([?]), and is due to Y. Hu. The "only if" part remains open.

Note that asking for a norm amounts to asking for a lattice: Given a norm $||\cdot||$, the unit ball is a lattice. Conversely, given a lattice Λ, its gauge $||x|| = q_C^{-v_\Lambda(x)}$, where $v_\Lambda(x) = \sup\{v \mid x \in \pi_C^v \Lambda\}$ is a n orm. Thus we are looking for integral structures in locally algebraic representations of G.

Also, requiring the completion with respect to this norm to be nonzero amounts to the lattice not containing any C-line.

Another remark - the condition that V arises from a potentially semistable Galois representation, translates (using Fontaine's weak admissiblity) to a very concrete criterion on V, which can be formulated for any reductive group G (Emerton's condition). Therefore it makes sense to consider arbitrary reductive groups.

3.1. **Progress so far.**

- For $GL_2(\mathbb{Q}_p)$, the work of Colmez [3] and Berger-Breuil [1]. (both use theory of (φ, Γ)-modules)
- Note that the central character of $BS(\rho)$ always attains values in $\mathcal{O}_C^\times$. Sorensen ([?]) has proved for any connected reductive group G defined over $\mathbb{Q}_p$, that if π_{alg} is an irreducible algebraic representation of $G(\mathbb{Q}_p)$, and π_{sm} is an essentially discrete series representation of $G(\mathbb{Q}_p)$, both defined over C, then $\pi_{alg} \otimes_C \pi_{sm}$ admits a $G(\mathbb{Q}_p)$-invariant norm if and only if its central character is unitary.
- The best results in the principal series case, which is the deepest, are by joint work of Caraiani, Emerton, Gee, Geraghty, Paskunas ans Shin ([?]). Using global methods, they construct a candidate $\Pi(\rho)$ for a p-adic local Langlands correspondence for $GL_n(F)$ and are able to say enough about it to prove new cases of the conjecture. Their conclusion is even somewhat stronger than the existence of a norm on $BS(\rho)$, in that it asserts admissibility.

Both works employ the usage of global methods, and as this is a question of local nature, we believe that there must be some local method to recover these results. There has also been some progress employing local methods, which yields results also for finite extensions of $\mathbb{Q}_p$, namely:

- For $GL_2(F)$, de Ieso ([?]), following the methods of Breuil for $\mathbb{Q}_p$, used compact induction together with the action of the spherical Hecke algebra to produce a separated lattice in $BS(\rho)$ where $BS(\rho)$ is an unramified locally algebraic principal series representation, under some technical p-smallness condition on the weight. This was later extended (A. !!! add reference) to lift some of the restrictions on the weight.
- For $GL_2(F)$, in a joint work with Kazhdan and de Shalit ([?]), we have used p-adic Fourier theory for the Kirillov model to get integral structures if $BS(\rho)$ is tamely ramified smooth principal series or unramified locally algebraic principal series.
- For general split reductive groups, Grosse-Klonne ([?]) looked at the universal module for the spherical Hecke algebra, and was able to show some cases of the conjecture for unramified principal series, again under some p-smallness condition on the Coxeter number (when $F = \mathbb{Q}_p$) plus other technical assumptions.
- For $GL_2(F)$, Vigneras (!!! add reference) introduced the method we will discuss today to obtain integral structures when $BS(\rho)$ is a tamely ramified smooth principal series.
- For $U_3(F)$, in (A. !!! add reference) we have genralized both Vigneras' and de Ieso's methods to obtain integral structures when $BS(\rho)$ is either a tamely ramified smooth principal series or unramified locally algebraic principal series.

Recently, Hu (!!! add reference) has used diagrams mod p to simplify the proofs of Vigneras. We shall generalize these ideas today.

4. Statement of the main result

4.1. **Notations.** Assume that $\mathbb{G}$ has semisimple rank 1, so one could think of GL_2 or U_3, for example, as special cases.

Let $\mathbb{B}$ be a minimal parabolic subgroup of $\mathbb{G}$, with Levi decomposition $\mathbb{B} = \mathbb{T}\mathbb{U}$, and let $B = \mathbb{B}(F)$, $U = \mathbb{U}(F)$.

let $\mathbb{S}$ be a maximal F-split torus in $\mathbb{T}$. Let $S = \mathbb{S}(F)$, $T = \mathbb{T}(F)$. Let $\mathbb{N} = N_{\mathbb{G}}(\mathbb{S})$, $N = \mathbb{N}(F)$, and let $W = N/T$ be the Weyl group.

Note that $|W| = 2$ by assumption, so we can write $W = \{1, w\}$. W acts on T by conjugation.

Let $\chi : T \to C^\times$ be a smooth character, which we inflate to a character $\chi : B \to C^\times$ via the quotient map $B \to T$. Let

$$Ind_B^G(\chi) = \{f \in C^\infty(G, C) \mid f(bg) = \chi(b)f(g) \quad \forall b \in B, g \in G\}$$

be the smooth parabolic induction with coefficients in C, such that G is acting by right translation: $(gf)(x) = f(xg)$.

We further denote by $Z(T)$ the center of T. We fix an open compact subgroup U_0 of U.
Let $Z(T)^+ = \{z \in Z(T)^+ : zU_0z^{-1} \subseteq U_0\}$ be the contracting monoid, and $\delta_B(t) = [U_0 : tU_0t^{-1}]$ be the modulus.
The action of W on T induces an action on χ, which we denote by χ^w.

Theorem 4.1. *(A.) Let $\Pi = Ind_B^G \chi$ be a smooth principal series C-representation. Assume that $\chi : T \to C^\times$ is tamely ramified. Assume that Π is irreducible. Then Π admits an integral structure if and only if $\left|\chi\mid_{Z(G)}\right| = 1$ and for any $z \in Z(T)^+$ we have*

$$|\chi(z)\delta_B(z)| \leq 1, \quad |\chi^w(z)| \leq 1$$

(This is Emerton's criterion)

Example 4.2. Let $G = GL_2(F)$, $S = T$ the torus of diagonal matrices, B the upper triangular Borel. Let

$$U_0 = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathcal{O}_F \right\}$$

Now $Z(T) = T$, and

$$Z(T)^+ = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mid v_F(t_1) \geq v_F(t_2) \right\}$$

In particular, up to the center we may assume $z \in Z(T)^+$ is of the form $\begin{pmatrix} t & \\ & 1 \end{pmatrix}$ with $t \in \mathcal{O}_F$. Then $t = u \cdot \varpi^a$, and

$$\chi(z) = \chi_1(u) \cdot \chi_1(\varpi)^a$$

hence $|\chi(z)| = |\chi_1(\varpi)|^a$. Also $\delta_B(z) = q^a$ Thus the conditions are

$$|\chi_1(\varpi)q|^a \leq 1, \quad |\chi_2(\varpi)|^a \leq 1$$

for all a. Which is equivalent simply to

$$|\chi_1(\varpi)q| \leq 1, \quad |\chi_2(\varpi)| \leq 1$$

i.e. $1 \leq |\chi_1(\varpi)| \leq \left|q^{-1}\right|$

!!!! Maybe add here the equivalence to being potentially semistable ??? !!! Look more closely at what should actually be the statement. Recall to address the gaps in functoriality.

5. Coefficient Systems on The Tree

5.1. **The Bruhat-Tits Tree of G.** We will recall a few facts about the Bruhat-Tits building, specializing to the case where G has semisimple rank 1.

Fact 5.1. *(Tits) The Bruhat-Tits building of G is 1-dimensional, i.e. it is a forest, which we denoe by $\mathcal{T}$. Moreover, $\mathcal{T}$ is bihomogenic of degrees $q^{d_0} + 1$, and $q^{d_1} + 1$, for some positive integers d_0, d_1. (!!! Check if I need to state the connection to K_0, K_1 !!!)*

Fact 5.2. *(BT) The group G acts on $\mathcal{T}$ via isometries, and its action on the vertices of $\mathcal{T}$ has at most two orbits.*

Fact 5.3. *(BT) The stabilizers of the vertices in $\mathcal{T}$ are maximal compact subgroups of G.*

Corollary 5.4. *There are at most two conjugacy classes of maximal compact subgroups in G.*

Fact 5.5. *(SS) For any simplex $x \in \mathcal{T}$, its stabilizer K_x admits a filtration by pro-p-groups $K_x(e)$ such that if $x \subseteq y$ we have $K_y(e) \subseteq K_x(e)$ for all $e \geq 1$ (!!! complete here the properties we will actually use !!!)*

5.2. **Coefficient systems.** Coefficient systems were introduced over $\mathbb{C}$ by [4]. In this section, we follow [**?**] and translate the language of coefficient systems to the group G. Let $\mathcal{T}$ be the Bruhat-Tits tree of G. Let R be a commutative ring.

Definition 5.6. An *R-coefficient system* $\mathcal{V} = \{V_\sigma\}_\sigma$ is a contravariant functor from the category of simplices in $\mathcal{T}$ (with inclusions as morphisms) to the category of R-modules.
Let $\mathcal{V} = (\{V_\sigma\}_{\sigma \subset \mathcal{T}}, \{r_\sigma^\tau\}_{\sigma \subset \tau})$ be an R-coefficient system on $\mathcal{T}$. We say that $\mathcal{V}$ is *G-equivariant* if for every $g \in G$ and every simplex $\sigma \subset \mathcal{T}$, we have linear maps $g_\sigma : V_\sigma \to V_{g\sigma}$ satisfying the following properties: (!!! maybe don't state them - just say the obvious properties !!!)

- For every $g, h \in G$ and every simplex $\sigma \subset \mathcal{T}$, we have $(gh)_\sigma = g_{h\sigma} \cdot h_\sigma$
- For every simplex $\sigma \subset \mathcal{T}$, we have $1_\sigma = id_{V_\sigma}$.

- For every $g \in G$ and every inclusion $\sigma \subset \tau$, the following diagram commutes:

$$\begin{tikzcd} V_\tau \arrow[r, "g_\tau"] \arrow[d, "r_\sigma^\tau"] & V_{g\tau} \arrow[d, "r_{g\sigma}^{g\tau}"] \\ V_\sigma \arrow[r, "g_\sigma"] & V_{g\sigma} \end{tikzcd}$$

We say that a G-equivariant R-coefficient system is an RG-coefficient system on $\mathcal{T}$.

Definition 5.7. Let $\mathcal{V}$ be an RG-coefficient system on $\mathcal{T}$. We define the complex of (oriented) chains with finite support $C_\bullet(\mathcal{V})$

$$C_i(\mathcal{V}) = \left\{ f : \coprod_{\sigma \in \mathcal{T}_i} \sigma \to \bigoplus_{\sigma \in \mathcal{T}_i} V_\sigma \mid f(\sigma) = 0 \quad a.e., f(s\sigma) = sgn(s) f(\sigma) \quad \forall s \in S_i \right\}$$

!!! Should say something about the difference between the two cases - in the first we can swap orientation, and in the second there is only one orientation !!!

There are obvious boundary maps $\partial : C_i(\mathcal{V}) \to C_{i-1}(\mathcal{V})$ and a natural G-action on $C_\bullet(\mathcal{V})$. The boundary maps give rise to homology $H_i(\mathcal{V}) = H_i(C_\bullet(\mathcal{V}))$.

Before we ssay why we care about coefficient systems, we would like to present a more compact way to encode them.

5.3. **Diagrams.** Using the transitivity of the G-action on the tree, we can encode a coefficient system by a diagram. There are two cases:

5.3.1. *G acts transitively on the vertices.* If this is the case, we let K_0 be the stabilizer of a vertex, and K_1 the stabilizer of an edge containing it. Then $K_1 = \langle I, t \rangle$ where $I = K_0 \cap K_1$ and t is an element which swaps v_0 and v_1. (Does it always exist? Yes, we can always send v_0 to v_1, but then v_1 can be sent to some other v_2. May then apply w.)

In this case, every RG-coefficient system is equivalent to the following data, which is called a "diagram" (Draw!!)

- A representation of K_1 on an R-module L_1 (= a representation of I, with an action of t)
- A representation of K_0 on an R-module L_0
- an $R[I]$-equivariant map $r : L_1 \to L_0$

The equivalence is given simply by $(r, L_0, L_1) \mapsto \left\{ g \cdot ind_{K_i}^G L_i \right\}_{g\sigma_i}$

This is the case for example, when $G = GL_2$, hence it is treated in Vigneras, Hu so we will focus on the other case.

Remark 5.8. The word "diagram" was introduced by Paskunas [**?**] in his construction of supersingular irreducible representations of $GL_2(F)$ on finite fields of characteristic p, and there is an equivalence of categories between $R[G]$-diagrams and G-equivariant coefficient systems on $\mathcal{T}$.

5.3.2. *G has two orbits on the vertices.* In this case, we let K_0, K_1 be the stabilizers of two adjacent vertices (which must represent the two orbits, because the G-action preserves the distance), and $I = K_0 \cap K_1$ is the stabilizer of the edge between them. (This time we have no element that swaps the vertices).

Then each G-equivariant coefficient system is, in fact, equivalent to a diagram (Draw!!)

Definition 5.9. Let R be a commutative ring. An $R[G]$-*diagram* consists of the following data:

- A representation of I on an R-module L_{01}.
- A representation of K_0 on an R-module L_0.
- A representation of K_1 on an R-module L_1.
- $R[I]$-equivariant maps $r_0 : L_{01} \to L_0$, $r_1 : L_{01} \to L_1$.

Again, the equivalence is given simply by induction - $ind_{K_i}^G L_i$.

We will refer to a diagram as a quintuple $(L_{01}, L_0, L_1, r_0, r_1)$, and depict such a diagram as

$$\begin{tikzcd} & L_0 \\ L_{01} \arrow[ur, "r_0"] \arrow[dr, "r_1"] & \\ & L_1 \end{tikzcd}$$

Corollary 5.10. *Let $\mathcal{L}$ be an RG-coefficient system on $\mathcal{T}$, equivlaent to the diagram $(L_{01}, L_0, L_1, r_0, r_1)$. Then*

$$C_0(\mathcal{L}) \cong ind_{K_0}^G L_0 \oplus ind_{K_1}^G L_1, \quad C_1(\mathcal{L}) \cong ind_I^G L_{01}$$

as RG-modules. The boundary map can be described as

$$\partial([1,v]) = ([1, r_0(v)], [1, -r_1(v)])$$

where $[g, v] \in ind_H^G V$ is the function supported on Hg^{-1} and attaining the value $v \in V$ at g^{-1}. Thus, we have an exact sequence of RG-modules

$$0 \to H_1(\mathcal{L}) \to ind_I^G L_{01} \to ind_{K_0}^G L_0 \oplus ind_{K_1}^G L_1 \to H_0(\mathcal{L}) \to 0$$

6. Vigneras' Zig-Zag Method

6.1. **The method.** What are coefficient systems good for? we have the following theorem.

Theorem 6.1. *(SS) Let V be a smooth C-representation of G. Let $x \in \mathcal{T}_0$ be a vertex, and let $e \geq 1$ be such that V is generated by its $K_x(e)$-invariants. Let $\gamma_e(V)$ be the coefficient system on $\mathcal{T}$ defined by $\gamma_e(V)(\sigma) = V^{K_\sigma(e)}$, with the obvious inclusion maps. Then $H_0(\gamma_e(V)) = V$.*

This theorem allows one (Vigneras) to work homologically on the tree for the purpose of constructing an integral structure. Together with the equivalence coefficient systems = diagrams, we get a convenient local criterion. (This also works for locally algebraic). Again, we assume to be in the case where G has two orbits on the vertices of $\mathcal{T}$.

Corollary 6.2. *(Vigneras, A.) The smooth C-representation V is $\mathcal{O}_C$-integral if and only if there exist integral structures L_0, L_1 of the representations of K_0, K_1 on $V^{K_0(e)}, V^{K_1(e)}$ respectively, such that $L_{01} := L_0 \cap V^{I(e)} = L_1 \cap V^{I(e)}$. In this case, if $\mathcal{L}$ is the coefficient system corresponding to (L_0, L_1, L_{01}) with inclusions, then $H_0(\mathcal{L})$ is an integral structure in V.*

Proof. (Sketch) By the thm, we have an exact sequence

$$0 \to H_1(\mathcal{V}) = \ker \partial \to ind_I^G \left(V^{I(e)}\right) \to ind_{K_0}^G \left(V^{K_0(e)}\right) \oplus ind_{K_1}^G \left(V^{K_1(e)}\right) \to V \to 0$$

Since r_0, r_1 are injective, $\ker \partial = 0$. We may now consider the long exact sequence of homology for $0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{V}/\mathcal{L} \to 0$. It yields

$$0 \to H_1(\mathcal{V}/\mathcal{L}) \to H_0(\mathcal{L}) \to V \to H_0(\mathcal{V}/\mathcal{L}) \to 0$$

Now, if $L_{01} = L_0 \cap V^{I(e)} = L_1 \cap V^{I(e)}$, then the maps $V^{I(e)}/L_{01} \to V^{K_i(e)}/L_i$ are injective, hence $H_1(\mathcal{V}/\mathcal{L}) = 0$, showing that $H_0(\mathcal{L}) \hookrightarrow V$. It is clearly a lattice, and one has to show it is separated, which follows from the discreteness of the valuation, working inductively on the support of chains. The converse is easy. $\square$

This criterion suggests a practical algorithm (The zig-zag method)

Definition 6.3. Let $M_0 \subseteq V^{K_0(e)}$ be a finitely generated $\mathcal{O}_C$-submodule. We may let

$$\mathcal{M}^{(0)} = (M_0^{(0)}, M_1^{(0)}, M_{01}^{(0)}) = (M_0, K_1 \cdot M_0^{I(e)}, M_0^{I(e)})$$

and then

$$\mathcal{M}^{(1)} = (M_0^{(1)}, M_1^{(1)}, M_{01}^{(1)}) = \left(K_0 \cdot \left(M_1^{(0)}\right)^{I(e)}, M_1^{(0)}, \left(M_1^{(0)}\right)^{I(e)}\right)$$

In general

$$\mathcal{M}^{(i)} = (M_0^{(i)}, M_1^{(i)}, M_{01}^{(i)})$$

with

$$M_{01}^{(i)} = \left(M_{i \mod 2}^{(i-1)}\right)^{I(e)}, M_{i \mod 2}^{(i)} = M_{i \mod 2}^{(i-1)}, M_{1-i \mod 2}^{(i)} = K_{1-i \mod 2} \cdot M_{01}^{(i)}$$

We say that $(\mathcal{M}^{(n)})_{n\geq 0}$ is the sequence of *zigzags* of M_0.

(Remark - the intersection is different each time - pulling back through a different inclusion. Also the choice of starting with M_0 is arbitrary. Could be M_1.)

(!!! Maybe we would like to present every step differently - check the proof of classification of diagrams mod p !!!!)

Corollary 6.4. *V has an integral structure iff the sequence of zigzags $(\mathcal{M}^{(n)})_{n\geq 0}$ is finite.*

This essentially follows from the fact that we are enlarging our finitely generated modules over the Noetherian ring $\mathcal{O}_C$.

6.2. **Application - Necessity of the criterion.** The first application of this method was due to Vigneras (2008) which was to prove a completely local criterion for the integrality of smooth tamely ramified principal series of $GL_2(F)$.

The second was (A. 2016) - proving a similar criterion for $U_3(F)$. Both proofs have an easy part - the necessity, and a difficult part - the sufficiency.

We describe here a sketch of the proof of the easy part.

Let $V = \mathrm{Ind}_B^G \chi$ be a smooth principal series, where $\chi : T \to C^\times$ is a tamely ramified character. Then χ is trivial on $T \cap I(1)$ and $e = 1$. Recall that $B = T \cdot U$.

Then its restriction to $T \cap I$ is the inflation of a character of the reduction $\overline{T} = T/T \cap I(1)$. (Note that $\overline{T}$ need not be the special fiber of the same reductive group as T is - that will only be true in an unramified case).

We also write $G_i = K_i/K_i(1)$, and let the image of $B \cap K_i, T \cap K_i$ under these maps be B_i, T_i, respectively. (Note that $T_0 \cong T_1 \cong \overline{T}$). Let χ_i be the reductions of χ. Let U_i be the unipotent radicals of B_i.

Denote by $red_i : K_i \to G_i$ the natural quotient maps for $i \in \{0, 1\}$.

Lemma 6.5. *The natural maps $(Ind_B^G \chi)^{K_i(1)} \to Ind_{B_i}^{G_i} \chi_i$ are isomorphisms, inducing an isomorphism $\left(Ind_B^G \chi\right)^{I(1)} \cong \left(Ind_{B_i}^{G_i} \chi_i\right)^{U_i}$, where $\chi_i = \chi \mid_{B \cap K_i}$, which, since χ is tamely ramified, factors through $K_i(1)$.*

Then the tamely ramified diagram

$$\begin{tikzcd}
 & \left(ind_B^G \chi\right)^{K_0(1)} \\
\left(ind_B^G \chi\right)^{I(1)} \arrow[ur] \arrow[dr] & \\
 & \left(ind_B^G \chi\right)^{K_1(1)}
\end{tikzcd}$$

is inflated from the inclusions

$$\left(ind_{B_i}^{G_i} \chi_i\right)^{U_i} \hookrightarrow ind_{B_i}^{G_i} \chi_i, \quad i \in \{0, 1\}$$

These lattices infla\`te to a tamely ramified diagram.

Let L_0 be the $\mathcal{O}_C$-integral structure of the C-representation of K_0 on $V_0 = \left(ind_{B \cap K_0}^{K_0} \chi_0\right)^{K_0(1)}$, given by functions with values in $\mathcal{O}_C$.

We will use it to start our zigzag algorithm.

It will be useful to have a basis for this $\mathcal{O}_C$-module.

For any $g \in K_0$ we denote by $f_g \in L_0$ the function supported on $(B \cap K_0) g K_0(1)$ with value 1 at g. By $K_0(1)$-invariance, we have $f_g = f_{gk}$ for all $k \in K_0(1)$.

We will identify between an element and its lift (a representative).

From the Bruhat decomposition for G_0 we have the following.

Proposition 6.6. *The $\mathcal{O}_C K_0$-module L_0 is cyclic, generated by f_1, $L_0 = \mathcal{O}_C K_0 \cdot f_1$. A basis for L_0 is given by $\{f_1, f_{wu}\}_{u \in U_0}$.*

Similarly, we would want a basis for $L_{01}^{(0)} = L_{01}^{I(1)}$.

Lemma 6.7. *A basis of $L_0^{I(1)}$ is given by $\phi_1 = f_1$ and $\phi_w = \sum_{u \in U_0} f_{wu}$.*

Before proceeding, there is another natural integral structure on V_1, which wil be useful in the sequel, so we investigate its properties.

Let L_1 be the $\mathcal{O}_C$-integral structure of the C-representation of K_1 on $V_1 = \left(ind_{B \cap K_1}^{K_1} \chi_1\right)^{K_1(1)}$, given by functions with values in $\mathcal{O}_C$.

For any $g \in K_1$ we denote by $h_g \in L_1$ the function supported on $(B \cap K_1) g K_1(1)$ with value 1 at g.

Here we have to choose some $t \in G$ such that $tv_1 = v_1$, but $tv_0 \neq v_0$. (note that $t \in Z(T)^+$, and in fact is of minimal valuation there).

Again, Bruhat decomposition gives us:

Proposition 6.8. *The $\mathcal{O}_C K_1$-module L_1 is cyclic, generated by h_{tw}, i.e. $L_1 = \mathcal{O}_C K_1 \cdot h_{tw}$. A basis for L_1 is given by $\{h_1, h_{twu}\}_{u \in U_1}$.*

In order to identify our modules with elements in this standard module, we have to see how ϕ_1, ϕ_w are expressed in terms of this basis.

Lemma 6.9. *We have*

$$\phi_1 = h_1 + \sum_{1 \neq u \in U_1} \chi(t \cdot w \cdot m(u)) \cdot h_{twu}, \quad \phi_w = \chi(t) \cdot h_{tw}$$

where here $m(u) \in Tw$ is a unique element associated to u in this coset. (comes from BT theory)

We begin by establishing necessity of one of the conditions.

Proposition 6.10. *Let $\chi : T \to C^\times$ be a tamely ramified character, such that $V = Ind_B^G \chi$ admits an integral structure. Let $z \in Z(T)$ such that $z U_{\alpha,0} z^{-1} \subseteq U_{\alpha,0}$. Then $|\chi^w(z)| \leq 1$.*

Proof. (Sketch) First, reduce to an element of minimal valuation. Thus, it suffices to prove that $|\chi^w(w(t))| = |\chi(t)| \leq 1$.

Proceeding with the zig-zag, we get

$$L_1^{'} = K_1 \cdot L_0^{I(1)} = K_1 \cdot (\mathcal{O}_C \cdot \phi_1 \oplus \mathcal{O}_C \cdot \phi_w) = \mathcal{O}_C K_1 \phi_1 + \chi(t) \cdot \mathcal{O}_C K_1 h_{tw}$$

By Proposition 6.8 we know that $L_1 = \mathcal{O}_C K_1 \cdot h_{tw}$. Recall that by Lemma 6.9, we have

$$\phi_1 = h_1 + \sum_{1 \neq u \in U_1} \chi(t \cdot w \cdot m(u)) \cdot h_{twu} \in L_1 = \mathcal{O}_C K_1 \cdot h_{tw}$$

It follows that

$$\chi(t) \cdot f_1 = \chi(t) \cdot \phi_1 \in \left(L_1^{'}\right)^{I(1)} \subseteq K_0 \cdot \left(L_1^{'}\right)^{I(1)} = z(L_0)$$

But by Proposition 6.6 we know that $L_0 = \mathcal{O}_C K_0 \cdot f_1$, hence $\chi(t) \cdot L_0 \subseteq z(L_0)$. By (!!!add cross-ref) the sequence of zigzags $(z^n(L_0))_{n \geq 0}$ is finite, hence $\chi(t) \in \mathcal{O}_C$ and we are done. □

This also yields the easiest case for sufficiency, namely:

Proposition 6.11. *Let $\chi : T \to C^\times$ be a tamely ramified character, such that for every $z \in Z(T)$ with $z U_0 z^{-1} \subseteq U_0$, we have$|\chi^w(z)| = 1$. Then $\Pi = Ind_B^G \chi$ admits an integral structure.*

Proof. Since $w(t) \in Z(T)$ is such that $w(t) U_{\alpha,0} w(t)^{-1} = U_{\alpha,2} \subseteq U_{\alpha,0}$, our assumptions imply that $|\chi(t)| = |\chi^w(w(t))| = 1$. It follows that $\chi(t) \mathcal{O}_C K_1 \cdot h_{tw} = \mathcal{O}_C K_1 \cdot h_{tw} = L_1$. Since $\phi_1 \in L_1$, it follows that $L_1^{'} = L_1$, so that by (!!! add cross-ref !!!) $\Pi = Ind_B^G \chi$ admits an integral structure. □

Lemma 6.12. *Let $w_1 = tw$. For any $1 \neq u \in U_1$, denote $H_u = w_1 u \cdot \phi_1 \in L_1^{'}$. Then $L_1^{'}$ is spanned over $\mathcal{O}_C$ by*

$$\{H_u\}_{u \in U_1}, \quad \phi_1, \quad \{\chi(t) \cdot h_{w_1 u}\}_{u \in U_1}, \quad \chi(t) \cdot h_1$$

Proof. Since $K_1/I \cong (K_1/K_1(1))/(I/K_1(1)) = G_1/B_1$, by Lemma **??** we see that

$$K_1 = I \coprod \left(\coprod_{u \in \overline{U}_{-1}} w_1 u \cdot I \right)$$

Therefore

$$L_1^{'} = \mathcal{O}_C K_1 \cdot \phi_1 + \chi(t) \cdot L_1 =$$

$$= \mathcal{O}_C I \cdot \phi_1 + \sum_{u \in \overline{U}_{-1}} \mathcal{O}_C w_1 u \cdot I \cdot \phi_1 + \chi(t) \cdot L_1 =$$

$$= \mathcal{O}_C \cdot \phi_1 + \sum_{u \in \overline{U}_{-1}} \mathcal{O}_C \cdot H_u + \chi(t) \cdot L_1$$

which, combined with Proposition 6.8 gives the desired result. □

Lemma 6.13. *Denoting $z(L_0) = K_0 \cdot L_1^{I(1)}$, we have*

$$q^{d_1} \cdot \chi^w(t) \cdot \phi_w = |U_1| \cdot \chi^w(t) \cdot \phi_w \in z(L_0)$$

Proof. (Sketch)

The idea is to directly compute

$$\sum_{u \in U_1} H_u = q^{d_1} \cdot \chi(w_1^2) \cdot h_{w_1} + \left(\sum_{1 \neq u \in U_1} \chi(w_1 \cdot m(u)) \right) \cdot h_1 +$$

$$+ \chi(w_1^2) \cdot \sum_{1 \neq v} \left(\sum_{u : \psi_u(1) \neq v} \chi(m(\psi_u^{-1}(v)) \cdot m(u \cdot \psi_u^{-1}(v)^{-1})) \right) h_{w_1 v}$$

and the next Lemma shows that this gives us what we wanted. □

Lemma 6.14. *Let $m \in Tw$. Then $\{m \cdot m(u)\}_{1 \neq u \in U}$ is a subgroup of $T \cap K_1$. Thus, if $\chi \mid_{T \cap K_1} \neq 1$. Then $\sum_{1 \neq u \in U_1} \chi(m \cdot m(u)) = 0$. Else, $\sum_{1 \neq u \in U_1} \chi(m \cdot m(u)) = |U_1| - 1 = q^{d_1} - 1$.*

Proof. Currently the proof is a very hideous case by case computation! If you have any idea how to prove it in this generality, it would be much appreciated. □

Next we compute the $\mathcal{O}_C K_0$-module M_0 generated by $\phi_w = \sum_{u \in U_0} f_{wu}$.

By a similar computation (maybe can unify them to a single Lemma) we have $F_u := uw \cdot \phi_w = f_1 + \sum_v \chi(\beta(u, v)) \cdot f_{wv}$, and M_0 is spanned by the $\{F_u\}$ and ϕ_w, and $\sum F_u = q^{d_0} f_1 + (q^? - 1)$

This shows the necessity of $\left| q^{d_1 + d_2} \chi^w(t) \right| \leq 1$. Since $q^{d_1 + d_2} = \delta_B(t)$, this is what we wanted.

This difficult part was originally established using a Fourier transform on the unipotent radical of the Borel. But now we have a new gadget - diagrams mod p. Let's use it.

7. Diagrams Mod p

Hu (add reference) has identified the potential of the diagrams mod p to simplify the original proof of Vigneras. We will generalize his ideas.

7.1. **Classification of diagrams mod p.**

Lemma 7.1. *Let D be a diagram of k-modules such that D_i is an admissible K_i-representation and r_0, r_1 are injective. Then $H_0(D) \neq 0$ and $H_1(D) = 0$.*

Proof. The first assertion follows from considering a chain supported on a single vertex. For the second one, note that injectivity of the r_i assures that ∂ is injective. □

Proposition 7.2. *Let $D = (D_{01}, D_0, D_1, r_0, r_1)$ be a diagram of k-modules, not necessarily finite dimensional, such that $H_0(D) = 0$. Then D has a filtration by sub-diagrams such that each graded piece has one of the following forms $(Q_{01}, Q_0, Q_1, q_0, q_1)$:*
(I0) $Q_{01} = k \cdot v$, $Q_0 = 0$, $q_0 = 0$, $Q_1 \cong ind_I^{K_1}(k \cdot v)$, q_1 is the natural map, where I acts on v via some character ψ.
(I1) $Q_{01} = k \cdot v$, $Q_1 = 0$, $q_1 = 0$, $Q_0 \cong ind_I^{K_0}(k \cdot v)$, q_0 is the natural map, where I acts on v via some character ψ. .
(II0) $Q_{01} = k \cdot v$, $Q_0 = 0$, $q_0 = 0$, where I acts on v via some character ψ, Q_1 is a quotient of $ind_I^{K_1}(k \cdot v)$ such that $\dim_k Q_1 \leq q^{d_1}$, possibly 0, q_1 is the natural map.
(II1) $Q_{01} = k \cdot v$, $Q_1 = 0$, $q_1 = 0$, where I acts on v via some character ψ, Q_0 is a quotient of $ind_I^{K_0}(k \cdot v)$ such that $\dim_k Q_0 \leq q^{d_0}$ possibly 0, q_0 is the natural map.
In particular, if D_{01} is of finite dimension, then

$$(q^{d_0} + 1) \cdot \dim_k D_1 + (q^{d_1} + 1) \cdot \dim_k D_0 \leq (q^{d_0} + 1)(q^{d_1} + 1) \cdot \dim_k D_{01}$$

and equality holds if and only if only diagrams of type (I0) or type (I1) appear as graded pieces of the filtration.

Definition 7.3. Let $D = (D_{01}, D_0, D_1, r_0, r_1)$ be a diagram of k-modules such that D_{01}, D_0 and D_1 are all finite dimensional. We say that D satisfies the *dimension relation* if there exist $m_0, m_1 \in \mathbb{Z}_{\geq 0}$ such that:

$$\dim_k D_{01} = m_0 + m_1, \quad \dim_k D_0 = m_0 \cdot (q^{d_0} + 1), \quad \dim_k D_1 = m_1 \cdot (q^{d_1} + 1)$$

We give some examples of diagrams which satisfy the dimension relation. For an absolutely irreducible k-representation σ of K_0, $\lambda \in k^\times$ and $\chi : U_1(F) \to k^\times$ a smooth character, we may denote:

$$\pi(\sigma, \lambda, \chi) := \left(\frac{ind_{K_0}^G \sigma}{T_0 - \lambda} \right) \otimes \chi \circ \det$$

where $T_0 \in End_G(ind_{K_0}^G \sigma)$ is the Hecke operator defined in [].

Example 7.4. Let $\pi = \pi(\sigma, \lambda, \chi)$ for some σ, λ, χ as above. Then the canonical diagram $D(\pi) := (D_{01}(\pi), D_0(\pi), D_1(\pi), \mathrm{can}_0, \mathrm{can}$
defined by

$$D_{01}(\pi) := \pi^{I(1)}, \quad D_0(\pi) := K_0 \cdot D_{01}(\pi) \subset \pi, \quad D_1(\pi) := K_1 \cdot D_{01}(\pi) \subset \pi$$
$$\mathrm{can}_0 : D_{01}(\pi) \hookrightarrow D_0(\pi), \quad \mathrm{can}_1 : D_{01}(\pi) \hookrightarrow D_1(\pi)$$

satisfies the dimension relation. In fact, one checks easily that $\dim_k D_{01}(\pi) = 2$, $\dim_k D_0(\pi) = q^{d_0} + 1$ and $\dim_k D_1(\pi) = q^{d_1} + 1$.
Note that the canonical diagram of $D(Sp)$ (resp. $D(1)$) of the Steinberg representation Sp (resp. the trivial representation 1) does not satisfy the dimension relation (but $D(\mathrm{Sp}) \oplus D(1)$ does).
Other examples of diagrams satifying the dimension relation are diagrams of type (I0) or (I1) in the above proposition ???. We give it a name for convenience.

Definition 7.5. A diagram $D = (D_{01}, D_0, D_1, r_0, r_1)$ of k-modules is said to be *naive* if it is of type (I0) or (I1) as in proposition ???.

By definition, if D is a naive diagram, then $\dim_k D_{01} = 1$, and either $D_0 = 0, \dim_k D_1 = q^{d_1}+1$, or $\dim_k D_0 = q^{d_0}+1$ and $D_1 = 0$. In both cases, D satisfies the dimension relation in Definition ???. (either with $d_0 = 0, d_1 = 1$ or with $d_0 = 1, d_1 = 0$ in the second case).

Lemma 7.6. (i) *If D is a naive diagram, then $H_0(D) = H_1(D) = 0$.*
(ii) Conversely, if $D = (D_{01}, D_0, D_1, r_0, r_1)$ is a diagram of k-modules such that $H_0(D) = H_1(D) = 0$, then D can be written as a successive extension of naive diagrams. In particular, if D_{01} is finite dimensional, then D satisfies the dimension relation.

7.2. **Application I (If there is time).** Applying it to a Steinberg representation.

7.3. **Application II - a criterion for the existence of integral structures.** Recall we have proven the necessity, and are left with sufficiency.

How do we do that?

Assume we have an infinite sequence of zigzags in $V_{01}, V_0, V_1 = V^{I(1)}, V^{K_0(1)}, V^{K_1(1)}$.

(!!! Not true !!!!)

Since V_0 is irreducible as as a K_0-representation (indeed, any subrepresentation would give rise to a subrepresentation of the reduction G_0, but by assumption this is irreducible), and since the coefficient field is discretely valued, there are only finitely many homothety classes of K_0-invariant $\mathcal{O}$-lattices in V_0. Therefore, there exist integers $n < n'$ such that $L_{01}^{(n)}$ and $L_{01}^{(n')}$ lie in the same homothety class, that is, there exists $\lambda \in C^\times$ such that

$$L_{01}^{(n')} = \lambda L_{01}^{(n)}$$

Since $L_0^{(n)}, L_1^{(n)}$ are generated from $L_{01}^{(n)}$, we get the same, and since $\mathcal{L}^{(n)} \subsetneq \mathcal{L}^{(n')}$, it follows that $val_L(\lambda) < 0$.

Let $\mathcal{L} = \mathcal{L}^{(n)}$. Since $H_0(\mathcal{L}) \to H_0(\lambda\mathcal{L})$ is surjective, we have $H_0(\lambda\mathcal{L}/\mathcal{L}) = 0$. By devissage, one deduces that $H_0(\varpi^{-1}\mathcal{L}/\mathcal{L}) = 0$. Equivalently, $H_0(\mathcal{L} \otimes_{\mathcal{O}} k) = 0$. By Proposition ???, since D satisfies the dimension relation, it must be a successive extension of naive diagrams.

Moreover, as $\dim_k D_{01} = 2$, it is the successive extension of exactly two such diagrams, one of each type.

Write $L_{01} = \mathcal{O} \cdot v_0 \oplus \mathcal{O} \cdot v_1$, where v_0, v_1 are C-linearly independent, and eigenvectors for the action of I, which acts as $\chi \mid_I$ on v_0 and as $\chi^w \mid_I$ on v_1.

Therefore, we may assume that $D_{01} = k \cdot \overline{v}_0 \oplus k \cdot \overline{v}_1$, with $D_0 = ind_I^{K_0}(k \cdot \overline{v}_0)$,$D_1 = ind_I^{K_1}(k \cdot \overline{v}_1)$, and $\overline{r}_0(\overline{v}_1) = 0$, $\overline{r}_1(\overline{v}_0) = 0$, with the action on v_0 given either by χ or by χ^w.

Assume first it is given by χ.

We then obtain that $r_0(v_1) \in \varpi L_0$, $r_1(v_0) \in \varpi L_1$.

Using Nakayama, we get $L_i = ind_I^{K_i}(\mathcal{O} \cdot v_i)$ (using the fact that χ, χ_1 are tamely ramified - "inflation"). In particular, L_0 has an $\mathcal{O}$-basis given by

$$v_0, \quad \{\kappa \cdot v_0\}_{\kappa \in K_0(1)/I(1)}$$

However, $\sum_{\kappa \in K_0(1)/I(1)} \kappa \cdot v_0$ is $I(1)$-invariant, and I acts on it via $\chi^w \mid_I$. It follows that that there exists some $\alpha \in \mathcal{O}$ such that

$$r_0(v_1) = \alpha \cdot \sum_{\kappa \in K_0(1)/I(1)} \kappa \cdot v_0$$

Since $r_0(v_1) \in \varpi L_0$, it follows that $\alpha \in \varpi\mathcal{O}$.

Similarly L_1 has an $\mathcal{O}$-basis given by v_1, $\{\kappa \cdot v_1\}_{\kappa \in K_1(1)/I(1)}$, with $\sum_{\kappa \in K_1(1)/I(1)} \kappa \cdot v_1$ being $I(1)$-invariant, and I acts on it via $\chi \mid_I$. It follows that

$$r_1(v_0) = \beta \cdot \sum_{\kappa \in K_1(1)/I(1)} \kappa \cdot v_1$$

for some $\beta \in \varpi\mathcal{O}$. However (by the above decription - make explicit), one obtains

$$\alpha \cdot \beta = \chi(t)$$

Therefore $|\chi(t)| < 1$, contradiction.

Explicitly, we start with $v_0 = a_0 \cdot \phi_1$ and $v_1 = a_1 \cdot \phi_w$, thus $r_0(v_1) = a_1 \cdot r_0(\phi_w) = a_1 \cdot \sum_u f_{su}$ and $r_1(v_0) = a_0 r_1(\phi_1) = a_0(h_1 + \sum_u \chi(f(u)) \cdot h_{twu})$

How are these connected? The first means $|a_1| < |a_0|$, and the second the converse so it can't be.

It follows we must have it the other way around, i.e. $r_0(\phi_1) = f_1$ and $r_1(\phi_w) = \chi^{-1}(t)$.

!!!! Wait !!!! Something is not right here. Think it over again in the morning with a clear head !!!

We have $\{f_1, f_{su}\}$ representing π^{K_0} and $\{h_t, h_1, h_{tu}\}$ representing π^{K_1}. This is with $\phi_1 = f_1$ and $\phi_s = \sum_u f_{su}$, while $\phi_s = t \cdot h_t$ and $\phi_1 = h_1 + \sum \chi(m(u)) \cdot h_{tu}$.

We also let $F_u := \sum_v sv \cdot f_{su}$ (? check this is the right summation) Then $\sum F_u$ gives us some nice multiple $\chi(z) \cdot f_1$ - shows necessity. Same with the h's.

By Hu's method we obtain a mod p diagram which is naive. It means that

7.4. **Further applications?** This approach might prove fruitful also for tamely ramified locally algebraic representations of low weight.

8. Computational Aspects

The nice thing about the zig-zag method is that it is very practical and down to earth. Since we are only studying existence of the integral structure, we may work with the universal completion, corresponding to the minimal commensurability class of integral structures - those of finite type. In particular, this means we are always working with finitely generated $\mathcal{O}$-modules, and after encoding the group action, all we need to worry about is basically linear algebra.˙

8.1. **Wild Ramification.** Theoretically, the zig-zag method could be applied also to higher ramification. The crux of it relies on Schneider-Stuhler, which is just as valid with higher ramification groups. Now, the coset representatives of interest are no longer cosets of the pro-p kernels $I(1), K_0(1), K_1(1)$, but the higher ramification groups $I(e), K_0(e), K_1(e)$.

The zig-zag no longer stabilizes after so few iterations, and calculations become more complicated.

Computations can indicate what we should expect (does it stabilize eventually? If so, after how many zig-zags? What is the resulting integral structure?

!!! Give example for $GL_2(F)$ with $e = 2$!!!

!!! See if I can get some code running an example !!!

8.2. **Non-smooth locally algberaic representations.** The zig-zag method is capable of proving the criterion for the existence of integral structure and finding it, when the representation is a smooth parabolic induction, or a locallly algebraic Steinberg representation. Diagrams mod p seem to offer a method for proving some of the cases (of low enough weight) of locally algebraic parabolic induction.

!!! See if I can get some results/code over here as well !!!!

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