The Canonical Model

Eran Assaf

Dartmouth College

Shimura Varieties Reading Seminar, July 2020
Model - analyst approach

Theorem (Shimura, Petri, ...)

If $\Gamma \subseteq PSL_2(\mathbb{Z})$ is a neat congruence subgroup, then the algebraic curve $X(\Gamma) = (\Gamma \backslash \mathcal{H})^*$ has a model defined over a number field.

Sketch of proof # 1.

$X(\Gamma)$ compact Riemann surface $\iff$ algebraic curve over $\mathbb{C}$.  
If $g > 1$, not hyperelliptic, $\omega_X$ very ample $\iff X \hookrightarrow \mathbb{P}^{g-1}$.  
But $\Gamma(X, \omega_X) = S_2(\Gamma)!$.  
If $f(\tau) = \sum_{n=1}^{\infty} a_n(f) q^n \in S_2(\Gamma)$ is an eigenform, $a_p(f)$ are eigenvalues of $T_p$, so all lie in a number field $K$.  
All relations $R_i(f_1, \ldots, f_g)$ defined over $K$ $\iff X(\mathbb{C}) = V(R_1, \ldots, R_m) = X_0(\mathbb{C})$, with $X_0/K$.  
\[\square\]
Model - geometer approach

Theorem (Shimura, Petri, ...)

If $\Gamma \subseteq PSL_2(\mathbb{Z})$ is a neat congruence subgroup, then the algebraic curve $X(\Gamma) = (\Gamma \backslash \mathcal{H})^*$ has a model defined over a number field.

Sketch of proof # 2.

$E$ elliptic curve, $H \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$, $\phi : E[N] \to \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. Let $(E, \phi) \sim_H (E', \phi') \iff \exists h \in H, \iota : E \sim E'$ s.t. $h \circ \phi = \phi' \circ \iota$

Then

- $S(H) = \{(E, \phi)\}/\sim_H$.
- $(E, \phi)^\sigma = (E^\sigma, \phi \circ \sigma^{-1})$ for $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- $(E, \phi)$ is $K$-rational iff $E/K$ and $\phi \circ \text{Gal}(\overline{\mathbb{Q}}/K) \circ \phi^{-1} \subseteq H$.
- $S(\Gamma)(\mathbb{C}) = \Gamma_H \backslash \mathcal{H}$.

Take maximal $K$ such that the above holds.
Model - the Shimura way

**Theorem (Shimura, [Shi67])**

If $F$ is totally real of degree $g$, $B/F$ a quaternion algebra s.t. $B \otimes \mathbb{R} \cong M_2(\mathbb{R}) \otimes \mathbb{H}^{g-1}$, $\mathfrak{o}$ maximal order, $\Gamma(\mathfrak{o}) = \mathfrak{o}^\times \cap B^+$. Then $X(\Gamma) = \Gamma(\mathfrak{o}) \backslash \mathcal{H}$ has a model over $H^+_F$.

**Remark**

- *This is only a special case of the theorem.*
- *When $g > 1$, this is not(!) the field of moduli of abelian varieties. Roughly - for any tot. imaginary quadratic $K/F$, there is a family $\Sigma_K$ of a.v. $A_x$ s.t. $B \otimes_F K \subseteq \text{End}_\mathbb{Q}(A_x)$. Each has a field of moduli $k_{\Sigma}$ which is not abelian over $F$, but over some $K'$. Then $C_F = \bigcap_{K'} C_{K'}$.***
- *Main point - models over $C_{K'} \implies$ model over $C_F$.*
- *Involves choices, but reciprocity laws uniquely determine it.*
Canonical model

Reminder - Connected components

Recall $\xrightarrow{1} G' \xrightarrow{\nu} G \xrightarrow{T} T \xrightarrow{1}$ induces $f_K : Sh_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K \to T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K)$ and if $Y = T(\mathbb{R})/T(\mathbb{R})^\dagger$, then

$$T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K) = T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/\nu(K) = Sh_{\nu(K)}(T, Y).$$

Moreover, it induces $\pi_K = \pi_0(Sh_K(G, X)) \cong Sh_{\nu(K)}(T, Y), Sh_K(G, X)^\circ = Sh'_K(G', X^+)$. 

Today

- Reflex field $E = E(G, X)$, algebraic number field.
- (Canonical) model $f_K : Sh_K(G, X)_0 \to (\pi_K)_0$ of $f_K$ over $E$.
- Reciprocity laws determining the model uniquely.
- $\text{Aut}(\mathbb{C}/E) \cup \pi_K \implies$ model of $Sh'_K(G', X^+)$ over $E'/E$. 
Models of varieties

Example (Elliptic Curve)

$E/\mathbb{C}$ elliptic curve has a model over $K$ iff $j(E) \in K$. Twists by different elements of $K^\times/(K^\times)^2$ give different models.

Existence - Why? [Pet17]

- Spreading.
  Algebraic $\iff$ defined over $k = k_0(\alpha_1, \ldots, \alpha_r)$, $k_0 = \mathbb{Q}[x_0]/P$. Replace $\alpha_j$ by $x_j$ to get a family over $S'$ with $k(S') = k$, and the fiber at $x_j = \alpha_j$ is $X$. Replace $S'$ by an open subset $S$ to get a smooth fibration over $\overline{\mathbb{Q}}$.

- Rigidity.
  Countably many Shimura varieties $\iff$ fibers of small deformations are isomorphic. But $S(\overline{\mathbb{Q}})$ is dense in $S$ (!).
Cocharacters

Definition (Conjugacy classes of cocharacters)

\[G/\mathbb{Q} \text{ reductive, } k \subseteq \mathbb{C}. \text{ Write} \]

\[\mathcal{C}(k) = G(k) \backslash \text{Hom}(\mathbb{G}_m, G_k).\]

Example (Unitary group)

Let \( G = U_K/\mathbb{Q}(2) \) for some quadratic extensions \( K/\mathbb{Q} \).

Let \( T = \left\{ \begin{pmatrix} a & 0 \\ 0 & \sigma(a)^{-1} \end{pmatrix} \bigg| a \in K^\times \right\} \). Then

\[X_*(T)_K = \left\{ t \mapsto \begin{pmatrix} t^m \sigma(t)^n & 0 \\ 0 & t^{-n} \sigma(t)^{-m} \end{pmatrix} \right\}_{m, n \in \mathbb{Z}}\]

and

\[X_*(T)_\mathbb{Q} = \left\{ t \mapsto \begin{pmatrix} \text{Nm}_{K/\mathbb{Q}}(t)^n & 0 \\ 0 & \text{Nm}_{K/\mathbb{Q}}(t)^{-n} \end{pmatrix} \right\}_{n \in \mathbb{Z}}\]
## Conjugacy classes of cocharacters

### Galois action

\[
\sigma(\text{Ad}(g) \circ \mu) = \text{Ad}(\sigma(g)) \circ \sigma(\mu) \quad \Rightarrow \quad \text{Aut}(k'/k) \circlearrowleft C(k')
\]

### Lemma

**Assume** \( G \) **splits** over \( k \). **Let** \( T \) **be a split** maximal torus. **Then**

\[
W \backslash \text{Hom}(\mathbb{G}_m, T_k) \rightarrow G(k) \backslash \text{Hom}(\mathbb{G}_m, G_k)
\]

**is bijective.** Here \( W = W(G_k, T) \) **is the Weyl group.**

### Proof.

All maximal split tori are conjugate \( \Rightarrow \) surjective. If \( \mu, \mu' \in X_*(T) \) are such that \( \mu = \text{Ad}(g) \circ \mu' \), let \( C = C_G(\mu(\mathbb{G}_m)) \). Then \( T \subseteq C \) and

\[
\text{Ad}(g)(T) \subseteq \text{Ad}(g)(C_G(\mu'(\mathbb{G}_m))) = C_G(\text{Ad}(g)\mu'(\mathbb{G}_m)) = C
\]

are maximal split tori in a connected reductive group \( C \). 

\( \exists c \in C(k) : \text{Ad}(cg)^{-1} T = T \quad \Rightarrow \quad cg \in N_G(T), \text{Ad}(cg) \circ \mu' = \mu. \)
Hodge cocharacter

**Definition (Hodge cocharacter)**

\((G, X)\) Shimura datum. The Hodge character is

\[ x \in X \quad \mapsto \quad h_x : \mathbb{S} \rightarrow G \quad \mapsto \quad h_x : \mathbb{S}_\mathbb{C} \rightarrow G_{\mathbb{C}} \]

\[ \mapsto \quad \mu_x : \mathbb{G}_m \xrightarrow{z \mapsto (z, 1)} \mathbb{G}_m \times \mathbb{G}_m \sim \mathbb{S}_\mathbb{C} \rightarrow G_{\mathbb{C}} \]

where \(\mathbb{S}_\mathbb{C} \sim \mathbb{G}_m \times \mathbb{G}_m\) is \(r \otimes z \mapsto (rz, r\bar{z})\).

**Example (Modular curve)**

Let \(G = GL_2(\mathbb{Q})\), \(X = \text{Ad}(G(\mathbb{R})) \cdot h\), \(h(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\).

Since \(\frac{z+1}{2} \otimes 1 - \frac{i(z-1)}{2} \otimes i \mapsto (z, 1)\), we have

\[ \mu_h(z) = \begin{pmatrix} \frac{z+1}{2} & -\frac{i(z-1)}{2} \\ \frac{i(z-1)}{2} & \frac{z+1}{2} \end{pmatrix} \]

\[ \text{Tr}(\mu_h(z)) = z + 1, \quad \det(\mu_h(z)) = z \]

Thus \(\mu_h(z) \sim \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\).
Reflex field - definition

Hodge cocharacter is algebraic

The Hodge cocharacters \( \{ \mu_x \}_{x \in X} \) define \( c(X) \in \mathcal{C}(\mathbb{C}) \). From the lemma

\[
\mathcal{C}(\overline{\mathbb{Q}}) \cong W(\overline{\mathbb{Q}}) \setminus \text{Hom}(\mathbb{G}_m, T_{\overline{\mathbb{Q}}}) = W(\mathbb{C}) \setminus \text{Hom}(\mathbb{G}_m, T_{\mathbb{C}}) \cong \mathcal{C}(\mathbb{C}).
\]

Definition (Reflex field)

\( E(G, X) \) is the field of definition of \( c(X) \) in \( \overline{\mathbb{Q}} \). Explicitly, if \( H_X = \text{Stab}_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} c(X) \), then \( E(G, X) = \overline{\mathbb{Q}}^{H_X} \).

Remark (Finiteness of the reflex field)

\( E(G, X) \subseteq k \), since \( W(k) = W(\overline{\mathbb{Q}}) \) and \( X_*(T)_k = X_*(T)_{\overline{\mathbb{Q}}} \).
Reflex field and Hodge cocharacters

**Lemma ([Kot84])**

If \( \mu \in c(X) \) is defined over \( k \), then \( E(G, X) \subseteq k \). If \( G \) is quasi-split over \( k \), and \( E(G, X) \subseteq k \), then \( c(X) \) contains a \( \mu \) defined over \( k \).

**Proof.**

\((\Rightarrow)\) If \( \sigma \in \text{Gal}(\bar{Q}/k) \) then \( \sigma(\mu) = \mu \), hence \( \sigma(c(X)) = c(X) \), so \( \text{Gal}(\bar{Q}/k) \subseteq H_X \). Taking fixed fields \( E(G, X) \subseteq k \).

\((\Leftarrow)\) S maximal \( k \)-split torus, \( T = C_G(S) \). \( G \) q.s. \( \implies T \) maximal torus. \( B \) a \( k \)-Borel containing \( T \), \( C \) the \( B \)-positive Weyl chamber of \( X_*(T) \otimes \mathbb{R} \). Since \( \bar{C} \) is a fundamental domain for \( W \), \( \exists \mu \in \bar{C} \cap c(X) \). If \( \sigma \in \text{Gal}(\bar{Q}/k) \), then \( \sigma \mu \in \bar{C} \) since \( B \) is a \( k \)-group, and since \( c(X) \) is fixed by \( \sigma \), \( \mu \), \( \sigma \mu \) are in the same \( W \)-orbit \( \implies \sigma \mu = \mu \).
Reflex fields of tori

Example (Torus)

$T$ torus over $\mathbb{Q}$, $h : S \to T_\mathbb{R}$. $\mu_h : \mathbb{G}_{m\mathbb{C}} \to T_\mathbb{C}$ is defined over $\bar{\mathbb{Q}}$, $E(T, \{h\})$ fixed field of the sub. of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ fixing $\mu_h \in X_*(T)$.

Example (CM Torus)

$(E, \Phi)$ CM type, $T = \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m$. $T(\mathbb{R}) = (E \otimes \mathbb{R})^\times \cong (\mathbb{C}^\Phi)^\times$.

Let $h_\Phi : S \to T_\mathbb{R}$ be s.t.

$$h_\Phi(\mathbb{R}) = z \mapsto (z, z, \ldots, z) : \mathbb{C}^\times \to (\mathbb{C}^\Phi)^\times.$$

On $\mathbb{C}$-points, $h_\Phi, \mathbb{C} : S_\mathbb{C} \to T_\mathbb{C}$ is the map

$$(z_1, z_2) \mapsto (z_1, \ldots, z_1, z_2, \ldots, z_2) : \mathbb{C} \times \mathbb{C} \to (\mathbb{C}^\Phi)^\times \times (\bar{\mathbb{C}}^\Phi)^\times.$$

The Hodge cocharacter is

$$\mu_\Phi = z \mapsto (z, \ldots, z, 1, \ldots, 1) : \mathbb{C}^\times \to (\mathbb{C}^\Phi)^\times \times (\bar{\mathbb{C}}^\Phi)^\times.$$

Thus $E(T, \{h_\Phi\})$ is the reflex field of $(E, \Phi)$. 

PEL Shimura varieties

Example (PEL Shimura varieties)

\((G, X)\) simple PEL datum of type (A) or (C). Then \(\sigma\) fixes the conjugacy class of \(h\) iff it fixes \(\text{Tr} \circ h\). Since \(B\) acts on \(V = W \otimes V_0\) via \(W\), case by case analysis shows that \(F_0(\{\text{Tr} \circ h(z)\}_{z \in \mathbb{C}}) = F_0(\{\text{Tr}_X(b)\}_{b \in B})\). Then \(E(G, X)\) is the field generated by \(\{\text{Tr}_X(b)\}_{b \in B}\).

Remark

Note that this is the field of definition of the moduli problem - \((A, i, s, \eta K)\) s.t. \(\text{A abelian variety, } \pm s \text{ polarization, } i : B \to \text{End}^0(A), \eta K \text{ level, } \text{Tr}(i(b)|T_0A) = \text{Tr}_X(b)\).
Shimura curves

**Example (Quaternion algebra)**

$F$ totally real, $B/F$ quaternions, $G = B^\times$, $I_{nc}$ the split places, $I_c$ the non-split places. Let $h(\mathbb{R}) : \mathbb{C}^\times \to \mathbb{H}^{I_c} \times GL_2(\mathbb{R})^{I_{nc}}$ be

$$a + bi \mapsto \left(1, 1, \ldots, 1, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \ldots, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right).$$

Then $c(X)$ contains $\mu : \mathbb{G}_{m\mathbb{C}} \to G_{\mathbb{C}}$,

$$z \mapsto (1, 1, \ldots, 1) \times \left(\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\right) \in GL_{I_c}^{2\mathbb{C}} \times GL_{I_{nc}}^{2\mathbb{C}},$$

so $E(G, X)$ is the fixed field of the subgroup stabilizing $I_{nc}$.

If $I_{nc} = \{v\}$, the case of Shimura curve, $E(G, X) = v(F)$. 
Example (Adjoint groups)

Assume $G$ adjoint. $T$ maximal torus in $G_{\bar{Q}}$, $\Delta$ a base for the roots of $\Phi(G, T)_{\bar{Q}}$. If $\sigma \in \text{Gal}(\bar{Q}/\mathbb{Q})$, it acts on $X^*(T)$, hence on $\Phi$, and $\sigma(\Delta)$ is also a base for $\Phi$. $W$ acts simply transitively on the bases of $\Phi$, so $\exists! w_\sigma \in W$ s.t. $w_\sigma(\sigma(\Delta)) = \Delta$. Then $\sigma \ast \alpha = w_\sigma(\sigma \alpha)$ is an action of $\text{Gal}(\bar{Q}/\mathbb{Q})$ on $\Delta$.

If $G$ is split, choose $T$ split and $B$ a $\mathbb{Q}$-Borel, so $\sigma(\Delta) = \Delta$, and the action is trivial. Since $G$ splits over a finite extension, the action is continuous.

Every $c \in C(\bar{Q})$ contains a unique $\mu : \mathbb{G}_m \to G_{\bar{Q}}$ such that $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \Delta$, because $W$ acts simply transitively on the chambers, and the map

$$c \mapsto (\langle \alpha, \mu \rangle)_{\alpha \in \Delta} : C(\bar{Q}) \to \mathbb{N}^\Delta$$

is bijective. Therefore $E(G, X)$ is the fixed field of the subgroup fixing $(\langle \alpha, \mu \rangle)_{\alpha \in \Delta} \in \mathbb{N}^\Delta$. Complex conjugation acts as $\alpha \mapsto -w_0 \alpha$. 
Special points

Definition (Special point)

$x \in X$ is special if $\exists T \subseteq G$ such that $h_x(\mathbb{C}^\times) \subseteq T(\mathbb{R})$. $(T, x)$ or $(T, h_x)$ is a special pair in $(G, X)$. When the weight is rational (SV4) and $Z(G)^\circ$ splits over a CM field (SV6), they are called CM-points and CM-pairs.

Remark

- If $(T, x)$ is special then $T(\mathbb{R})$ fixes $x$. If $T$ maximal and fixes $x$, $h(\mathbb{C}^\times) \subseteq C_G(T)(\mathbb{R}) = T(\mathbb{R})$, so $x$ is special.
- $Z(G)^\circ$ is isogenous to $G^{ab} \quad \implies \quad$ (SV6) means $G^{ab}$ splitting over a CM field. With (SV2), $G$ splits over a CM field. (SV4) shows $h_x : \mathbb{S} \to T$ factors through $G_{Hod}$, and as it factors through a CM torus, it also factors through the Serre group, so for any rep. $(V, \rho)$ of $T$, $(V, \rho \circ h_x)$ is the Hodge structure of a CM motive.
Special points and CM

Example (Modular curve)

\[ G = \text{GL}_2, \ X = \mathcal{H}^\pm = \mathbb{C}\backslash\mathbb{R}. \] Let \( z \in \mathbb{C}\backslash\mathbb{R} \) is s.t. \( E = \mathbb{Q}[z] \) is quadratic imaginary. Embed \( E \hookrightarrow M_2(\mathbb{Q}) \) using the basis \( \{1, -z\} \), to get a maximal subtorus \( \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m \subseteq G \). The kernel of \( e \otimes z \mapsto z : E \otimes \mathbb{C} \to \mathbb{C} \) is spanned by \( z \otimes 1 + 1 \otimes (-z) \), which is \( (z, 1) \) in our basis, representing \( z \). The map is \( E \otimes \mathbb{R} \)-linear, hence \( (E \otimes \mathbb{R})^\times \) fixes \( z \). Thus \( z \) is special. If \( z \) is special, it is fixed by some \( t \in G(\mathbb{Q}) \), so \( \mathbb{Q}[z] \) is quadratic.

Remark

If \( \text{Sh}_K(G, X) = \{(A, \ldots)\}/\sim \), special points are a.v.s of CM-type. The theory of CM describes how an open subgroup of Aut(\( \mathbb{C} \)) acts on such points. Shimura constructs an action on the special points, that agrees with it.
### The homomorphism $r_x$

**Definition (action on special points)**

Let $T$ be a torus over $\mathbb{Q}$, $\mu \in X_*(T)$ defined over $E$. For $Q \in T(E)$, let

$$\sum_{\rho: E \to \bar{\mathbb{Q}}} \rho(Q) \in T(\bar{\mathbb{Q}})$$

be stable under the Galois action, so is in $T(\mathbb{Q})$. Let $r(T, \mu) : \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m \to T$ be such that

$$r(T, \mu)(P) = \sum_{\rho: E \to \bar{\mathbb{Q}}} \rho(\mu(P)) \quad \forall P \in E^\times.$$

Let $(T, x)$ be a special pair, $E(x)$ the field of definition of $\mu_x$. Then

$$r_x : \mathbb{A}_E^\times \to T(\mathbb{A}_\mathbb{Q}) \to T(\mathbb{A}_{\mathbb{Q},f}).$$

Explicitly, if $a = (a_\infty, a_f) \in \mathbb{A}_E^\times$, then

$$r_x(a) = \sum_{\rho: E \to \bar{\mathbb{Q}}} \rho(\mu_x(a_f)).$$
Shimura reciprocity

Definition (Canonical model)

A model $M_K(G, X)$ of $\text{Sh}_K(G, X)$ over $E(G, X)$ is canonical if for every special pair $(T, x)$ and $a \in G(\mathbb{A}_f)$, $[x, a]$ has coordinates in $E(x)^{ab}$ and

$$\sigma[x, a]_K = [x, r_x(s)a]_K$$

for all $\sigma \in \text{Gal}(E(x)^{ab}/E(x))$, $s \in \mathbb{A}_{E(x)}^\infty$ with $\text{art}_{E(x)}(s) = \sigma$.

Example (Tori)

$T$ torus over $\mathbb{Q}$, $h : S \rightarrow T_\mathbb{R}$. $(T, h)$ is a Shimura datum, $E = E(T, h)$ is the field of definition of $\mu_h$. $\text{Sh}_K(T, H)$ is a finite set, and Shimura reciprocity defines a continuous action of $\text{Gal}(E^{ab}/E)$ on $\text{Sh}_k(T, h)$. This defines a model of $\text{Sh}_K(T, h)$ over $E$, which is, by definition, canonical.
### Complex multiplication

**Example (CM tori)**

$(E, \Phi)$ CM-type. $(T, h_\Phi)$ as before. Then $E(T, h_\Phi) = E^*$, and $r(T, \mu_\Phi) : \text{Res}_{E^*/Q} \mathbb{G}_m \to \text{Res}_{E/Q} \mathbb{G}_m$ is given by

$$r(T, \mu_\Phi)(z) = \prod_{\rho : E^* \to \mathbb{Q}} \rho(z, z, \ldots, z, 1, 1, \ldots, 1)$$

and we want $z' \in E$ mapping to it. However, recall that $V = E \otimes \mathbb{R} \simeq \mathbb{C}^\Phi$ is stable by $\text{Aut}(\mathbb{C}/E^*)$, hence there is an $E^*$-vector space $V_0$ such that $V = V_0 \otimes \mathbb{C}$. $z \in E^*$ acts on $V$ via multiplication by $\text{Nm}_{E^*/Q}(z)$, which on $\mathbb{C}^\Phi$ is $\text{Nm}_{E^*/Q}(z) \times 1_\Phi$. But as an $E$-vector space, this is a line, so there is $z' \in E$ such that this is multiplication by $z'$. This is how we defined the reflex norm $N_{\Phi^*}(z)$. Therefore $r(T, \mu_\Phi) = N_{\Phi^*}$. 
CM tori as moduli spaces

**Lemma (CM tori)**

\[ \text{Sh}_K( T, h_\Phi) \text{ classifies triples } (A, i, \eta K) \text{ s.t. } A \text{ is a.v. of CM type } (E, \Phi) \text{ and } \eta : V(\mathbb{A}_f) \rightarrow V_f(A) \text{ is an } E \otimes \mathbb{A}_f \text{-linear isomorphism}. \]

**Proof.**

\[ \dim_E V = 1. \]  
\[ E \text{-action on } V \text{ gives } T \hookrightarrow GL_\mathbb{Q}(V). \]  
If \((A, i)\) is of CM-type \((E, \Phi)\), there is \(a : H_1(A, \mathbb{Q}) \rightarrow V\) carrying \(i_A\) to \(i_\Phi\). The isomorphism \(a \circ \eta : V(\mathbb{A}_f) \rightarrow V(\mathbb{A}_f) \) is \(E \otimes \mathbb{A}_f\)-linear, so is multiplication by \(g \in (E \otimes \mathbb{A}_f)^\times = T(\mathbb{A}_f)\). The map \((A, i, \eta) \mapsto [g]\) is the bijection.

**Galois action**

Same as classes over \(\overline{\mathbb{Q}}\). Let \(\mathcal{M}_K\) be the set of such triples. Then \(\text{Gal}(\overline{\mathbb{Q}}/E^*) \) acts on \(\mathcal{M}_K\), via \(\sigma(A, i, \eta K) = (\sigma A, \sigma \circ i, \sigma \circ \eta)\).
Reciprocity law and CM

Proposition (Reciprocity law = CM)

The map \((A, i, \eta) \mapsto [a \circ \eta]_K : \mathcal{M}_K \to \text{Sh}_K(T, h_\Phi)\) commutes with the Galois actions on both sides.

Proof.

Main theorem of CM \(\iff\) \(E\)-linear \(\alpha : A \to \sigma A\) s.t. \(\alpha(N_{\Phi^*}(s) \cdot x) = \sigma x\) for \(x \in V_f(A)\), where \(s \in \mathbb{A}_{E^*}\) is such that \(\text{art}_{E^*}(s) = \sigma|E^*\). Then \(a \circ V_f(\alpha)^{-1} : V_f(\sigma A) \to V(\mathbb{A}_f)\) is an \(E\)-isomorphism, and so

\[
\sigma(A, i, \eta) = (\sigma A, \sigma \circ i, \sigma \circ \eta) \mapsto [a \circ V_f(\alpha)^{-1} \circ \sigma \circ \eta]_K.
\]

But \(V_f(\alpha)^{-1} \circ \sigma = (N_{\Phi^*})_f(s) = r_{h_\Phi}(s)\), so

\[
[a \circ V_f(\alpha)^{-1} \circ \sigma \circ \eta]_K = [r_{h_\Phi}(s) \cdot (a \circ \eta)]_K.
\]
Robert E Kottwitz. 
Shimura varieties and twisted orbital integrals. 

Chris Peters. 
Rigidity of spreadings and fields of definition. 

Goro Shimura. 
Construction of class fields and zeta functions of algebraic curves. 