THE p-ADIC LOCAL LANGLANDS PROGRAMME AND INVARIANT NORMS

Abstract. Since the statement of the Taniyama-Shimura-Weil conjecture, it has been known that this is only the tip of the iceberg regarding the deep connection between solving equations in integers, and modular forms. The modularity theorem, proved in 2001, is only a special case of the famous Langlands conjecture for the group $GL_2(\mathbb{Q})$, which to this day remains open. Establishing the local langlands correspondence, in 2001, for $GL_n(F)$, where F is a local field, was not enough. The aim of the p-adic Local Langlands Programme is to establish a similar correspondence of representations with p-adic coefficients. This turns out to be quite involved, and was established for $GL_2(\mathbb{Q}_p)$ only recently. One of the main tools in establishing this correspondence was the existence of $GL_2(\mathbb{Q}_p)$-invariant norms in certain representations of $GL_2(\mathbb{Q}_p)$. We prove a criterion for the existence of such norms in certain representations of $U_3(F)$, where F is a finite extension of $\mathbb{Q}_p$.

1. Introduction

1.1. **Self introduction.** Hello, my name is Eran Assaf, a PhD student here in the Einstein institute, under the guidance of Prof. de Shalit.

I would like, first, to thank the Institute and prof. Zafriri's family for this generous prize, and for the opportunity to speak here today.

I will talk about the p-adic Langlands programme, and my research which lies in its framework, but let us begin with some background and motivation.

1.2. **Diophantine Equations.** The problem of solving equations in integral unknowns has occupied mankind for many years, the first evidence dating back to the Babylonians, during the reign of the Hammurabi dynasty (1800-1600 BC), who have considered the equation

$$x^2 + y^2 = z^2$$

later known as the Pythagorean equation, with its solutions considered as "Pythagorean triples".

This problem was given a full solution already in Euclid's Elements (Book X, ~300 BC), but the problem and its solution were by means of geometric algebra, as the abstract methods were not yet developed.

The greek Diophantus, who lived during the 2nd century AD, had set the stepping stone for abstract algebra by introducing notations for variables, and using these ideas, and the idea of rational parametrization, he performed a complete investigation of quadratic equations in two variables, in fact inspiring our current understanding of the genus 0 case.

He was also the first, at least documented, to have studied equations of the form

$$y^2 = f(x)$$

where $f(x)$ is any cubic polynomial. These remain a mystery to these very days!

1.3. **Elliptic Curves.** The equations of the above sort, when $f(x)$ is separable, are called elliptic curves, as they arose later, during the 18th century, while trying to calculate the arclengths of an ellipse, forming integrals of the form $\int \frac{dx}{y}$, with the defining equation as above. The understanding of the elliptic curve as a geometric object when x, y are complex variables, opened the door to a renewed investigation of its rational points, that is rational solutions.

It turns out that the points on the elliptic curve form an abelian group (this is in fact the simplest example of an Abelian variety), and this was used to establish many results regarding those curves.

The first step in this direction was the Mordell theorem (1922), stating that if E is an elliptic curve defined over $\mathbb{Q}$ (i.e. the coefficients of $f(x)$ are rational), then the group of its rational points $E(\mathbb{Q})$ is finitely generated.

Howevere, there is more to it than that. In order to determine the rank of this group, which is a primary point of interest, very complicated methods are needed, to which we will give some motivation in what follows.

Ever since the groundbreaking works of Riemann and Dirichlet in the theory of numbers, it has been known that investigating problems in number theory could be converted to questions in complex analysis by introducing a tool called L functions.

1.3.1. *Example - the Riemann zeta function.* Riemann (1859) considered the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$$

extended its domain by letting s be a complex variable.

It is immediate that the series converges for all s with $\Re(s) > 1$. By considering the following Mellin transform, which is just a version of the Fourier transform w.r.t. the multiplicative Haar measure $\frac{dx}{x}$, one can see that $\zeta(s)$ is the Mellin transform of the theta function $\phi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$, with $\theta(t) = 2\phi(t) + 1$, i.e.

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} \phi(t) t^{\frac{s}{2}-1} dt \tag{1.1}$$

where $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$ is the Gamma function.

Now, the function ϕ has a rapid decay for large values of $|t|$, and also satisfies the identity

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

This allows us to calculate the above integral also for all other values of s, hence obtain a meromorphic continuation to the entire complex plane of Λ, hence also of the zeta function ζ, from which we obtain a functional equation $\Lambda(s) = \Lambda(1-s)$.

At last, a thorough examination of the zeroes of the zeta function, and its behaviour in the so called "critical strip" - $0 < \Re(s) < 1$, has paramount importance, and, using its Euler product, which reveals its behaviour at different primes, yields as a consequence the prime number theorem.

We will try to generalize this idea of defining a series with an Euler product, showing that it is the tranform of some "nice" theta function, and in this way one could obtain a meromorphic continuation and a functional equation, which will prove useful in investigation of our problem.

The idea is that the Euler product of the L function encodes the information we have about the behaviour with respect to the powers of any given prime. In that sense, we may form an L function associated with an elliptic curve E, defined over $\mathbb{Q}$, also known as the Hasse-Weil zeta function.

When searching for solutions in $\mathbb{Q}_p$, there are some primes where the object in question is no longer an elliptic curve. When reduction mod p gives us an elliptic curve, we say have a good reduction at this prime. Otherwise, we say that this is a prime of bad reduction. There are always only finitely many primes of bad reduction, namely the ones dividing the discriminant Δ.

With each elliptic curve, we may associate a number N, called the conductor of E, measuring how "bad" is the bad reduction. In general, this is just some power of Δ, the discriminant of $f(x)$.

For a prime p with good reduction, we let $a_p = p + 1 - N_p$, where $N_p = |E(\mathbb{F}_p)|$ is the number of points mod p on the curve, and we let

$$L_p(E, s) = \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1}$$

When p is a prime of bad reduction, there is also a definition of $L_p(E, s)$.

[1]We may then define

$$L(E, s) = \prod_p L_p(E, s)$$

This product converges for $\Re(s) > 3/2$ only. Hasse conjectured (1954) that this function admits analytic continuation to the whole complex plane, and satisfies a functional equation, relating, for any s, $L(E, s)$ and $L(E, 2-s)$. This conjecture was finally verified as a consequence of the proof of the Taniyama-Shimura(1957)-Weil (1967) conjecture, which we will state in what follows.

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[1]For a prime p with multiplicative reduction, i.e. when the reduction of E to $\mathbb{F}_p$ yields a node (a double point), that is the case when $p \mid N$ and $p^2 \nmid N$, we define

$$L_p(E, s) = (1 \pm p^{-s})^{-1}$$

depeneding whether the multiplicative reduction is split or not, i.e. the slopes of the tangents to the node are in $\mathbb{F}_p$. (minus sign for split)

When p has additive reduction, we let $L_p(E, s) = 1$.

1.3.2. *What is it good for? or the Birch Swinnerton-Dyer conjecture.* As the Hasse conjecture holds, it makes sense to speak of the value of the above L function at $s = 1$. A remarkable conjecture made by Birch and Swinnerton-Dyer (1965) states that the order of vanishing of the L function at $s = 1$ equals the rank of $E(\mathbb{Q})$, and predicts the leading term of the Taylor series $L(E, s)$ at that point in terms of several quantities attached to the elliptic curve.

Currently, we know that if $L(E, 1) \neq 0$, then $rank(E(\mathbb{Q})) = 0$, and that if L has a first order zero at $s = 1$, then $rank(E(\mathbb{Q})) = 1$ (Gross-Zagier 1986, Kolvyagin 1989 + modularity Theorem).[2]

Nothing has been proved for curves with rank greater than 1, although there is extensive numerical evidence for the truth of the conjecture.

2. The Langlands programme

In order to understand the method for proving Hasse's conjecture, we must now follow a different path, which developed during the early 20th century, which is the study of modular forms.

2.1. **Modular forms.** Let $\mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ be the upper half plane. It is equipped with a natural action of $GL_2(\mathbb{R})^+$, namely

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$$

Let $\Gamma \leq GL_2(\mathbb{R})^+$ be a discrete subgroup. Then it acts properly discontinuously on $\mathcal{H}$, and one may form the quotient space $\Gamma\backslash\mathcal{H}$. It is usually non-compact, but by adding a finite number of points lying on $\partial\mathcal{H}$, we may compactify it. Those points are named *cusps*.

The simplest example is of course $\Gamma = SL_2(\mathbb{Z})$, where $\Gamma\backslash\mathcal{H}$ can be viewed as the fundamental domain (draw!) and the only cusp is at infinity.

A *modular form of weight* k with respect to a discrete subgroup $\Gamma \leq GL_2(\mathbb{R})^+$ is a complex-valued function $f : \mathcal{H} \to \mathbb{C}$, satisfying the following 3 conditions:

(1) f is a holomorphic function on $\mathcal{H}$.

(2) for any $z \in \mathcal{H}$, and any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, one has

$$f(\gamma z) = (cz + d)^k f(z)$$

(3) f is holomorphic at the cusps. [3]

Let

$$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}$$

Then a modular form with respect to $\Gamma_0(N)$ is called a modular form of *level* N.

For such a modular form, as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, we see that $f(z+1) = f(z)$, hence, as f is holomorphic at ∞, f admits a Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n q^n$, where $q = e^{2\pi i z}$.

When f vanishes at all cusps, we say that f is a *cusp form*. In particular, for cusp forms, $a_0 = 0$.

Modular forms w.r.t. $SL_2(\mathbb{Z})$ can be viewed as functions on lattices by setting $F(\mathbb{Z}+\tau\mathbb{Z}) = f(\tau)$, and $F(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$, when $\Im(\omega_1/\omega_2) > 0$.

Similarly, one may introduce a notion of a level on lattices, and view modular forms w.r.t $\Gamma_0(N)$ as functions on lattices of level N.

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[2] If anyone asks about the torsion, the conjecture was extended to include the prediction of the precise leading Taylot coefficient of the L function at $s = 1$. It is given by

$$\frac{L^{(r)}(E, 1)}{r!} = \frac{\#Sha(E)\Omega_E R_E \prod_{p|N} c_p}{(\#E_{tor})^2}$$

where $Sha(E)$ is the Tate-Shafarevich group (which is only conjectured to be finite), R_E is the elliptic regulator - $r \times r$ determinant whose matrix entries are given by a height pairing applied to a system of generators of E/E_{tor}, the c_p are elementary local factors $((E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p))$, where E_0 is the lift of the nonsingular points), and Ω_E is a simple multiple (1 or 2) of the real period of E (that is $\int_{E(\mathbb{R})} |\omega|$, where $\omega = \frac{dx}{2y}$).

[3] If anyone asks:Any cusp is of the form $\alpha \cdot \infty$, for some $\alpha \in SL_2(\mathbb{Z})$. There exists some $h \in \mathbb{N}$ such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \alpha^{-1}\Gamma_0(N)\alpha$, hence, if $f_\alpha(z) = (cz + d)^{-k} f(\alpha z)$ we have $f_\alpha(z + h) = f_\alpha(z)$, so that the condition of holomorphicity at the cusps is that f_α has a Fourier expansion $f_\alpha(z) = \sum_{n=0}^{\infty} b_n q^{n/h}$, where $q = e^{2\pi i z}$. We say that f vanishes at the cusp $\alpha \cdot \infty$ if $b_0 = 0$.

From now on, let us fix N.

Hecke (1937) introduced certain natural operators on these functions, which are given by summing over all lattices of a certain index, that is

$$T_n F(\Lambda) = n^{k-1} \sum_{[\Lambda:\Lambda']=n} F(\Lambda')$$

It turns out that these operators commute for m, n coprime, and are diagonalizable whenever $(n, N) = 1$.

As the space $S_k(N)$ of cusp forms of weight k and level N is finite dimensional, one may consider forms which are eigenvectors for all the T_p, $p \nmid N$ a prime. We call them *eigenforms*. If f is an eigenform for the primes $(p \nmid N)$ which is new (i.e. not coming from any $N' \mid N$), normalized such that $a_1 = 1$, we have a multiplicativity property - $a_m a_n = a_{mn}$ for $(m, n) = 1$, which urges us to define an L series by setting

$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

Due to the multiplicative property, one has an Euler product

$$L(f, s) = \prod_p L_p(f, s) = \prod_{p \nmid N} (1 - a_p p^{-s} + p^{k-1-2s})^{-1} \cdot \prod_{p \mid N} (1 - a_p p^{-s})^{-1}$$

which now looks familiar. In particular, for $k = 2$, one obtains an expression similar to the L function of the elliptic curve.

Moreover, in this case, we can show holomorphic continuation of the L series, and functional equation, due to the properties of f. This happens since

$$\int_0^{\infty} f(iy) y^{s-1} dy = (2\pi)^{-s} \Gamma(s) L(f, s)$$

and the integral converges for $\Re(s) > \frac{k}{2} + 1$, and the modularity of f yields a functional equation, allowing us to extend the above function to the entire complex plane.

It follows that all we need to show is that for any elliptic curve E, there exists some modular form of weight 2, f, such that $L(E, s) = L(f, s)$. Then we could show that it has analytic continuation and functional equation, allowing us to speak about its value at $s = 1$, as we have always wanted.

This was the Taniyama-Shimura-Weil conjecture, now known as the modularity theorem, proved in 2001 by Breuil, Conrad, Diamond and Taylor, by completing the work of Wiles.

2.2. **Automorphic forms.** This is beautiful, but this just the beginning. The modularity theorem is only a special case of a much larger design. In order to understand it, we would like to consider a different point of view on modular forms.

For this, we require the notion of the Adele ring, $\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p$, where the restricted product means that for an adele $a = (a_\infty, a_2, a_3, \ldots)$, all but a finite number of the a_ps are p-adic integers.

We may then talk about $GL_2(\mathbb{A})$. Let $K_p = GL_2(\mathbb{Z}_p)$, and $K_f = \prod_p K_p$. By the strong approximation theorem, one has

$$GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) \cdot GL_2^+(\mathbb{R}) \cdot K_0(N)$$

where $K_0(N)$ is the subgroup of K_f made of matrices whose lower left entry is divisible by N.

Let f be a modular form of weight k and level N. The adelization of f is the function $\varphi_f : GL_2(\mathbb{A}) \to \mathbb{C}$ defined, for $g = \gamma h_\infty k$, by

$$\varphi_f(g) = \det(h_\infty)^{k/2} (ci + d)^{-k} f(h_\infty \cdot i)$$

where $h_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This is well defined, φ_f is continuous, and $\varphi_f(\gamma g) = \varphi_f(g)$ for any $\gamma \in GL_2(\mathbb{Q})$ and $g \in GL_2(\mathbb{A})$. The function φ_f is, in fact, an automorphic form on $GL_2(\mathbb{A})$, which means that is satisfies some additional finitenes, growth, and, when f is a cusp form, a cuspidality condition. Also, the scalar matrices $Z(\mathbb{A})$ act trivially.

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[4]If anyone asks:

A lattice is called even if $|z|^2$ is an integer for all $z \in L$, and an even lattice is of level N, if N is the minimal such that $\sqrt{N} L^\sharp$ is even. Here $L^\sharp = \{x \in \mathbb{C} \mid \Re(x\overline{y}) \in \mathbb{Z} \quad \forall y \in L\}$.

[5]It turns out that for f cuspidal, such functions φ_f are bounded, hence square integrable on $Z(\mathbb{A})\cdot GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$, and one may form the space of such functions to obtain an Hilbert space, denoted by $L_0^2(Z(\mathbb{A})\cdot GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$.

The function φ_f, in turn, gives rise to a unitary automorphic representation of $GL_2(\mathbb{A})$, by taking the minimal closed subspace of $L_0^2(Z(\mathbb{A})\cdot GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$ containing φ_f, which will be denoted by π_f.

It turns out, (this is the tensor product theorem), that π_f could be written as

$$\pi_f = \widehat{\bigotimes_p}\pi_{f,p}$$

where $\pi_{f,p}$ is a smooth admissible irreducible representation of $GL_2(\mathbb{Q}_p)$.

Here, smooth means that every vector has an open stabilizer, and admissible means that the space of fixed vectors by any compact open is finite dimensional.

Then, for any automorphic representation $\pi = \widehat{\bigotimes}_p \pi_p$, one could construct an L series $L(\pi_p, s)$ for any p, and let $L(\pi, s) = \prod_p L(\pi_p, s)$ in such a way that $L(\pi_{f,p}, s) = L_p(f, s)$.

On the other hand, the L series of an elliptic curve could be viewed as an L series associated with a certain representation of the Galois group $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on a two dimensional space over $\mathbb{Q}_l$ for some prime l.

More precisely, one may consider the group of algebraic points on E- $E(\overline{\mathbb{Q}})$. It admits an action of $G_{\mathbb{Q}}$ and one may consider the l-torsion points $E[l] = \{P \in E \mid l\cdot P = 0\}$. When $(l, N) = 1$, one has an isomorphism $E[l] \simeq \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$. Considering a good prime l, one may form the inverse limit $\varprojlim_n E[l^n] \simeq \mathbb{Z}_l \times \mathbb{Z}_l$, which is still equipped with a Galois action of $G_{\mathbb{Q}}$, and after scalar extension, we obtain a representation $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}_l})$. For any such representation, one may form an L function (constructed by Artin), $L(\rho, s)$, and when $\rho = \rho_E$ is the representation coming from the elliptic curve, one has $L(\rho_E, s) = L(E, s)$. Its component at p is associated with the restriction of the representation to $G_p = Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

It follows that the modularity theorem shows a connection between the L series associated with 2-dimensional Galois representations on the one hand, and those associated with smooth, admissible, irreducible, unitary automorphic representations of $GL_2(\mathbb{A})$. This correspondence (in a more general context) is the conjectured Langlands correspondence for $GL_2(\mathbb{Q})$. (known as the Langlands' reciprocity conjecture).

Quite more generally, one might consider arbitrary global fields instead of $\mathbb{Q}$, and other reductive groups instead of GL_2. Then the correspondence is no longer one to one, and on the Galois side we do not have exactly representations, but things we refer to as Langlands parameters.
[6]

2.3. **Local Langlands correspondence.** As noted before, since the corresponding L functions have local factors at each prime p, one should expect some correspondence between the local factors.

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[5]If anyone asks - these are the following:

1. $GL_2(\mathbb{Q})$-left invariance
2. K-finiteness - for $k_\infty = \rho_\theta \in SO_2(\mathbb{R})$, $k_f \in K_0(N)$, $g \in GL_2(\mathbb{A})$, one has $\varphi_f(gk_\infty k_f) = e^{2\pi i k\theta}\varphi_f(g)$, where $k \in \mathbb{Z}$ is the weight of f. This means that the subspace $Span(\{g\cdot\varphi_f \mid g \in K\})$ is finite dimensional.
3. $\mathfrak{z}$-finiteness - One has the differential equation $\Delta\varphi_f = \frac{k}{2}(1-\frac{k}{2})\varphi_f$, where the Casimir operator Δ acts on the infinite component. This implies that φ_f is Δ-finite. Together with the next item, it implies that it is $\mathfrak{z}$-finite.
4. Action of the center - for any $z \in Z(\mathbb{A})$, $g \in GL_2(\mathbb{A})$, $\varphi_f(zg) = \varphi_f(g)$.
5. Growth condition - For any norm $||\cdot||$ on $GL_2(\mathbb{A})$, there exists a real number $A > 0$ such that $\varphi_f(g) \ll ||g||^A$. In other words, φ_f is of moderate growth. It is simpler to prove that if f is a cusp form, φ_f is actually bounded.
6. Cuspidality - If f is a cusp form, then φ_f is cuspidal, in the sense that for any $g \in GL_2(\mathbb{A})$, one has

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

7. It is important to notice that since φ_f is bounded for cuspidal f, $|\varphi_f|$ is square integrable on $Z(\mathbb{A})\cdot GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$.

[6]More about it:

The above representations turn out to be irreducible for almost all l, and when they are irreducible, they correspond to cuspidal representations - this yields the modularity theorem.

The general conjecture is still open, though for example, the Langlands-Tunnel theorem states that when the image of ρ in $PGL_2(\mathbb{C})$ is solvable, it holds.

Also, if one wants to be exact, the Galois side consists of representations of the Weil group (and not Galois representations).

This sort of correspondence, between n-dimensional representations of G_p on the Galois side, and certain irreducible unitary representations of $GL_n(\mathbb{Q}_p)$ on the automorphic side, is the Local Langlands correspondence. This correspondence was established when the coefficient field is either $\mathbb{Q}_l$ ($l \neq p$) or $\mathbb{C}$, by Harris and Taylor (2001), and Henniart (2000), independently.

However, we still fall short of a proof for the global conjecture.

A question which one should ask himself naturally is: "what happens when $l = p$?"

3. The p-adic Langlands programme

3.1. **Overview.** The original aim of the local p-adic Langlands programme is to look for a possible p-adic analogue of the classical and l-adic correspondence. From now on, we fix C - a finite extension of $\mathbb{Q}_p$.

All our representations will be vector spaces over C, which is p-adic. (not complex !).

The local p-adic correspondence for $GL_2(\mathbb{Q}_p)$ was fully developed, essentially by Berger, Breuil and Colmez (2010) using the theory of (φ, Γ)-modules. It was only completed in 2014 for the cases $p < 5$.

For some representations on the Galois side, called "potentially semistable", it is possible to attach a smooth representation $\pi_{sm}(\rho)$ on the automorphic side, as in the classical case. However, $\rho \rightsquigarrow \pi_{sm}(\rho)$ is no longer reversible. However, as the coeffiicient field is now an extension of $\mathbb{Q}_p$, we may also construct irreducible algebraic representation $\pi_{alg}(\rho)$ of $GL_2(\mathbb{Q}_p)$ from some data associated to ρ.

Still, one cannot reconstruct ρ from $\pi_{sm}(\rho)$ and $\pi_{alg}(\rho)$.

The problem is that these certain ρ are classified by some linear algebra data which includes a filtration, which is lost when constructing the $\pi_{alg}(\rho)$ and the $\pi_{sm}(\rho)$.

Note that as the coefficient field is p-adic, these two representations "live" in the same universe, and it makes sense to consider the representation $\pi_{sm}(\rho) \otimes \pi_{alg}(\rho)$.

These representations are no longer smooth, neither are they algebraic, but they are locally algebraic, meaning that each vector has an open neighbourhood, in which the G-action is polynomial.

The p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ takes any continuous representation $\rho : Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(C)$ and attaches a Banach C-space $\Pi(\rho)$ with a unitary $GL_2(\mathbb{Q}_p)$-action. This map $\rho \rightsquigarrow \Pi(\rho)$ is reversible, and compatible with classical local Langlands in the following sense: When ρ is potentially semistable,

$$\Pi(\rho)^{alg} = \pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$$

Furthermore, $\Pi(\rho)^{alg} = 0$ otherwise.

Here, the superscript alg indicates taking the locally algebraic vectors, i.e. vectors on which some compact open subgroup acts polynomially.

This means that the existence of locally algebraic vectors is equivalent (!) to the fact that our representation comes from a "good" Galois representation, representing some diophantine problem.

When ρ is irreducible, $\Pi(\rho)$ is known to be the completion of $\pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$ relative to a suitable $GL_2(\mathbb{Q}_p)$-invariant norm $\|\cdot\|$ which somehow corresponds to the lost filtration.

3.2. **The Breuil Schneider conjecture and integral structures.** For groups other than $GL_2(\mathbb{Q}_p)$ very little is known. One of the main conjectures was stated in 2007 by Breuil and Schneider, and in some sense it is a "first approximation" - for certain $\rho : Gal(\overline{\mathbb{Q}}_p/F) \to GL_n(C)$, one can define the representation $BS(\rho) := \pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$, and if it is indeed a subrepresentation of some larger, conjectural, Banach C-space $\Pi(\rho)$ with a unitary $GL_n(F)$-invariant norm, it should admit an invariant $GL_n(F)$-invariant norm. The resulting completions should be closely related to the yet undefined $\Pi(\rho)$ - at least in the irreducible cases.

We say that a representation V is locally algebraic if $V = V^{alg}$.

Conjecture 1. *(Breuil, Schneider [?]) The representation V arises from a potentially semistable Galois representation if and only if V admits a $GL_n(F)$-invariant norm.*

The "if" part is completely known for $GL_n(F)$ (2009), and is due to Y. Hu. The "only if" part remains open.

By the classification of irreducible representations of a reductive group G, there are three main types of representations to consider, called supersingular, discrete series and principal series representations. The case of supersingular representations is trivial. Let us briefly survey the results regarding the other two types.

3.3. **Progress on the BS conjecture.**

- Note that the central character of $BS(\rho)$ always attains values in $\mathcal{O}_C^\times$. Sorensen (2013) has proved for any connected reductive group G defined over $\mathbb{Q}_p$, that if π_{alg} is an irreducible algebraic representation of $G(\mathbb{Q}_p)$, and π_{sm} is an essentially discrete series representation of $G(\mathbb{Q}_p)$, both defined over C, then $\pi_{alg} \otimes_C \pi_{sm}$ admits a $G(\mathbb{Q}_p)$-invariant norm if and only if its central character is unitary.
- Recently there has been spectacular progress on the BS conjecture in the principal series case, which is the deepest, by joint work of Caraiani, Emerton, Gee, Geraghty, Paskunas ans Shin (2013). Using global methods, they construct a candidate $\Pi(\rho)$ for a p-adic local Langlands correspondence for $GL_n(F)$ and are able to say enough about it to prove new cases of the conjecture. Their conclusion is even somewhat stronger than the existence of a norm on $BS(\rho)$, in that it asserts admissibility.

Both works employ the usage of global methods, and as this is a question of local nature, we believe that there must be some local method to recover these results. There has also been some progress employing local methods, which yields results also for a finite extension, F, of $\mathbb{Q}_p$, namely:

- For $GL_2(F)$, Vigneras (2008) used homological methods on the Bruhat-Tits tree, which we extend and employ ourselves, to produce integral structures if ρ is a tamely ramified smooth principal series.
- For $GL_2(F)$, de Ieso (2013), following the methods of Breuil for $\mathbb{Q}_p$, used compact induction together with the action of the spherical Hecke algebra to produce a separated lattice in $BS(\rho)$ where ρ is an unramified locally algebraic principal series representation, under some technical p-smallness condition on the weight. In his thesis, de Ieso also shows that these lattices form an admissible completion.
- For $GL_2(F)$, in a joint work with Kazhdan and de Shalit (2013), we have used p-adic Fourier theory for the Kirillov model to get integral structures if ρ is tamely ramified smooth principal series or unramified locally algebraic principal series.
- For general split reductive groups, Grosse-Klonne (2014) looked at the universal module for the spherical Hecke algebra, and was able to show some cases of the conjecture for unramified principal series, again under some p-smallness condition on the Coxeter number (when $F = \mathbb{Q}_p$) plus other technical assumptions.

3.4. **The case of $G = U_3(F)$.**

As many attempts were made in order to find criteria for the existence of integral structures in representations of $GL_2(F)$, where F is a finite extension of $\mathbb{Q}_p$, and towards the proof of the Breuil-Schneider conjecture, which concerns the case of $GL_n(F)$, and somewhat more generally, the case of split reductive groups, very little is known about the correspondence for non-split reductive groups, and the unitary group, in particular.

We restrict ourselves only to the case of a smooth tamely ramified principal series representation, and give a necessary and sufficient criterion for the existence of such a norm.

This makes use of the method of coefficient systems on the Bruhat-Tits building, which is a tree for U_3, introduced by Vigneras in (2008).

This is still a work in progress, as we currently extend the methods of Breuil and de Ieso to obtain similar results for the unramified locally algebraic principal series representations.

4. Statement of the main result

We begin by introducing some notations -

4.1. **Notations.**

Let F be a finite extension of $\mathbb{Q}_p$, E a quadratic extension of F, and V a 3 dimensional vector space over E.

Let $\sigma \in Gal(E/F)$ be the nontrivial involution. We shall often denote $\overline{x} = \sigma(x)$ for $x \in E$.

$\mathcal{O}_E$ is the ring of integers in E, π a uniformizer, and q the cardinality of the residue field.

We shall denote by E^1 the norm one elements in E, i.e. $E^1 = U_1(F) = \{x \in E \mid x\overline{x} = 1\}$.

We shall further denote by C, as before, a finite extension of $\mathbb{Q}_p$, and by $\mathcal{O}_C$ its ring of integers.

Denote by θ the an Hermitian form on E^3, and let

$$G = U_3(F) = U(\theta) = \{g \in GL_3(E) \mid {}^t\overline{g}\theta g = \theta\}$$

be the unitary group, that is all the invertible linear transformations on V which preserve θ.

In order to state the results, we will also need the notion of a principal series representation.

Definition 2. (**principal series representaions**) Let $\chi : E^\times \to C^\times$, $\chi_1 : E^1 \to C^\times$ be multiplicative characters. The *principal series* representation $PS(\chi, \chi_1)$ is defined as

$$PS(\chi,\chi_1) = \left\{ f : G \to C \mid f\left(\begin{pmatrix} t & * & * \\ 0 & s & * \\ 0 & 0 & \bar{t}^{-1} \end{pmatrix} \cdot g \right) = \chi(t)\chi_1(s) \cdot f(g), \quad f \quad smooth \right\}$$

This space is equipped with a natural G-action by right translation, explicitly $(gf)(x) = f(xg)$. The functions are smooth with respect to this action.

4.2. **Main Result.**

Theorem 3. *(A.) Let $\chi : E^\times \to C^\times$, $\chi_1 : E^1 \to C^\times$ be tamely ramified characters. Then*

$$1 \leq |\chi(\pi)| \leq |q^{-2}| \iff PS(\chi, \chi_1) \quad admits\ a\ G\text{-}invariant\ norm$$

This should verify a generalization of the Breuil-Schneider conjecture for this representation of G, as the above condition should translate to being potentially semistable on the Galois side. However, as our method of proof does not reveal explicitly the norm, it does not give us a candidate for the correspondence for U_3, namely the completion of $PS(\chi, \chi_1)$ w.r.t. to this norm.

Note that the field C is p-adic, not complex!

4.2.1. *Locally algebraic representations.* This is a work in progress, and we are currently working on the locally algebraic case, and expect to have in a short time a criterion for the existence of integral structures when $\chi \otimes \chi_1$ is unramified, and under some smallness condition on ρ. Furthermore, in the case of unramified characters we shall be able to show that the resulting completion is admissible, namely

Definition 4. (admissible unitary Banach representations) Let B be a unitary Banach representation of G. Denote by B^0 its unit ball with respect to an invariant norm. Then B is said to be admissible if $B^0 \otimes_{\mathcal{O}_C} k_C$ is admissible in the sense of smooth represntations over k_C, i.e. its subspace of invariant elements under any open compact subgroup of G is finite dimensional. This definition is independent of the choice of B^0.