Functoriality of quantization: a KK-theoretic approach

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Literature

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Paths that cross

1. **Symplectic geometry and geometric quantization:**

   Guillemin–Sternberg (-Dirac) conjecture $[Q, R_G] = 0$

   'Geometric quantization commutes with symplectic reduction'

   Reformulation in terms of equivariant index theory (Bott)

   Defined and proved for compact groups and manifolds

2. **Operator algebras and equivariant K-theory:**

   Baum–Connes conjecture $\mu_r : K_\bullet^G(\underline{E}G) \xrightarrow{\cong} K_\bullet(C_r^*(G))$

   Interesting for noncompact groups G **(and proper actions)**

3. **Functoriality of quantization**

   Can symplectic data 'neatly' be mapped into operator data?

   Are geometric and deformation quantization perhaps related?

The Janus faces of quantization

1. Heisenberg (1925): classical observables $\rightsquigarrow$ matrices
2. Schrödinger (1926): classical states $\rightsquigarrow$ wave functions
3. von Neumann (1932): unification through Hilbert space
   matrices $\rightarrow$ operators, wave functions $\rightarrow$ vectors

1. Classical observables form Poisson algebra: commutative algebra/$\mathbb{C}$ and also Lie algebra with Leibniz rule $[fg, h] = f[g, h] + [f, h]g$

   Quantum observables form C*-algebra $\Rightarrow$ first face:

   Deformation quantization: Poisson algebra $\rightsquigarrow$ C*-algebra
2. Classical states form Symplectic manifold (M, ω) $\Rightarrow$ 2nd face:

   Geometric quantization: symplectic manifold $\rightsquigarrow$ Hilbert space

Symplectic form makes $C^\infty(M)$ Poisson algebra w.r.t. $fg(x) = f(x)g(x)$ and $\{f, g\} = \omega(\xi_f, \xi_g)$, Hamiltonian vector field ξ_f: $\omega(\xi_f, \eta) = \eta f$

Poisson manifold has Poisson algebra structure on $C^\infty(M)$, *ibid.*

Key examples of quantization

1. **Deformation quantization (Rieffel)**

   Lie group G, Lie algebra $\mathfrak{g}$, Poisson mfd $\mathfrak{g}^*$: for $X \in \mathfrak{g}$,
   $\hat{X} \in C^\infty(\mathfrak{g}^*)$ **defined by** $\hat{X}(\theta) = \theta(X)$, $\{\hat{X}, \hat{Y}\} = \widehat{[X, Y]}$

   Quantization of Poisson algebra $C^\infty(\mathfrak{g}^*)$ is C*-algebra $C^*(G)$

2. **Traditional geometric quantization (Kostant, Souriau)**

   compact symplectic manifold (M, ω) such that $[\omega] \in H^2(M, \mathbb{Z})$

   $\Rightarrow$ **$\mathbb{C}$-line bundle $L \to M$ plus connection ∇^L with $F(\nabla^L) = 2\pi i\omega$**

   $\Rightarrow$ **almost complex structure J s.t. $g(\xi, \eta) = \omega(\xi, J\eta)$ is metric**

   $\Rightarrow$ **Hilbert space** $Q(M, \omega, J) = \{s \in \Gamma(L) \mid \nabla^L_{J\xi - i\xi} s = 0, \xi \in \mathbf{X}(M)\}$

3. **Postmodern geometric quantization (Bott)**

   $Q_B(M, \omega, J)$ **is integer** $\text{index}(\not{D}^L) := \dim(\ker(\not{D}^L_+)) - \dim(\ker(\not{D}^L_-))$

   $\not{D}^L$ **is Spin^c Dirac operator on M defined by J coupled to L**

Guillemin-Sternberg conjecture

$G \circlearrowleft M$: **Lie group action on symplectic** M **s.t. momentum map**

$\Phi : M \to \mathfrak{g}^*$, $\Phi_X(x) = \langle \Phi(x), X \rangle$ **yields** $\mathfrak{g}$**-action:** $\xi_X^M = \xi_{\Phi_X}$ $(X \in \mathfrak{g})$

1. **Symplectic reduction:** $M/\!/G = \Phi^{-1}(0)/G$ **has symplectic form** ω_G
2. **Geometric quantization:** $Q(M/\!/G, \omega_G)$ **exist if** $Q(M, \omega)$ **exists;**

   $\exists$ **line bundle** $(L/\!/G \to M/\!/G, \nabla^{L/\!/G})$ **with** $F(\nabla^{L/\!/G}) = 2\pi i \omega_G$

'Quantization commutes with reduction': is this the same as

1. **Equivariant unitary geometric quantization** $G \circlearrowleft Q(M, \omega)$ **through**

   Kostant's formula $Xs = (\nabla^L_{\xi_X^M} - 2\pi i \Phi_X)s$, $s \in Q(M, \omega) \subset \Gamma(L)$
2. **'Quantum reduction':** $Q(M, \omega)/\!/G = Q(M, \omega)^G$ **(Dirac) ?**

In other words: $\mathbf{Q}(\mathbf{M}/\!/\mathbf{G}, \omega_\mathbf{G}) \overset{?}{\cong} \mathbf{Q}(\mathbf{M}, \omega)^\mathbf{G}$ **(as Hilbert spaces)**

Proved for M **compact Kähler and** G **compact by Guillemin&Sternberg**

More general symplectic manifolds require reformulation

Guillemin-Sternberg-Bott conjecture

1. **Symplectic reduction:** $(M/\!/G, \omega_G)$, **same as before**
2. **Bott's geometric quantization:** $Q_B(M/\!/G, \omega_G) = \mathrm{index}(\not{D}^{L/\!/G})$

Bott's reformulation of G-S conjecture: is this the same as

1. **Equivariant geometric quantization (**$G\&M$ **compact!):**
   $Q_B(M, \omega) = \mathrm{index}_G(\not{D}^L) = [\ker(\not{D}^L_+)] - [\ker(\not{D}^L_-)] \in R(G)$
2. **Quantum reduction:** $Q_B(M, \omega)^G$, $([V]-[W])^G = \dim(V^G) - \dim(W^G)$**?**

In other words, G-S-B conjecture: $(\mathrm{index}_{\mathbf{G}}(\not{\mathbf{D}}^{\mathbf{L}}))^{\mathbf{G}} = \mathrm{index}(\not{\mathbf{D}}^{\mathbf{L}/\!/\mathbf{G}})$

- **Proved by many people in mid 1990s (Meinrenken, . . .)**

For noncompact G **and** M **need substantial reformulation of G-S-B conjecture, under assumptions:** $G \circlearrowright M$ **proper and** M/G **compact**

Noncompact groups and manifolds

Compact $\rightsquigarrow$ **noncompact dictionary (suggested by Baum-Connes):**

- **Representation ring** $R(G) \rightsquigarrow K_0(C^*(G)) \cong KK_0(\mathbb{C}, C^*(G))$
- $\mathrm{index}_G(\not{D}) \in R(G) \rightsquigarrow \mu_M^G([\not{D}]) \in K_0(C^*(G))$ **N.B.** $C^*(G)$ **not** $C_r^*(G)$**!**

$[\not{D}] \in K_0^G(M) \cong KK_0^G(C_0(M), \mathbb{C})$ **equivariant K-homology of** M

$\mu_M^G : K_0^G(M) \to K_0(C^*(G))$ **'unreduced' analytic assembly map (Bunke)**

- **Quantum reduction** $R(G) \to \mathbb{Z} \rightsquigarrow K_0(C^*(G)) \xrightarrow{x \mapsto x^G} K_0(\mathbb{C}) \cong \mathbb{Z}$

  induced by map $C^*(G) \to \mathbb{C}$ **determined by trivial rep of** G

$\Rightarrow$ **Generalized G-S-B conjecture:** $\mu_M^G\left(\left[\not{D}^L\right]\right)^G = \mathrm{index}\left(\not{D}^{L/\!/G}\right)$

Proved by Hochs-Landsman (2008) if G **contains cocompact discrete normal subgroup, general proof by Mathai-Zhang (2010)**

Epilogue: functorial quantization

- 'Explains' generalized Guillemin-Sternberg-Bott conjecture as a special instance of functoriality of quantization
- Unifies the Janus faces of quantization into a functor Q

1. Domain of Q: Weinstein's category of (quantizable) 'dual pairs'
   (a) (integrable) Poisson manifolds as objects
   (b) (regular) symplectic bimodules $[P_1 \leftarrow M \rightarrow P_2]_{\cong}$ as arrows
2. Codomain of Q: Kasparov's category KK_0
   (a) C*-algebras as objects
   (b) [Graded Hilbert bimodules $A \circlearrowright \mathcal{E} \circlearrowleft B$ with $\not{D}]_h$ as arrows
3. Hypothetical quantization functor (based on examples only)
   (a) Deformation quantization: $P_i \rightsquigarrow$ C*-algebra A_i
   (b) Geometric quantization: $M \rightsquigarrow$ "[Spin^c Dirac operator $\not{D}^L]_?$"
   (c) Functorial quantization: $P_1 \leftarrow M \rightarrow P_2 \rightsquigarrow [\not{D}^L] \in KK_0(A_1, A_2)$

Examples of functorial quantization

1. **Symplectic manifold M yields dual pair $pt \leftarrow M \rightarrow pt$**

   (a) **Deformation quantization:** $pt \rightsquigarrow \mathbb{C}$

   (b) **Geometric quantization:** $(M,\omega) \rightsquigarrow [\not{D}^L]_?$

   (c) **Functorial quantization:** $(pt \leftarrow M \rightarrow pt) \rightsquigarrow [\not{D}^L] \in KK_0(\mathbb{C},\mathbb{C})$

   Identification $KK_0(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$ **identifies** $[\not{D}^L] \cong \mathrm{index}(\not{D}^L)$

2. **Hamiltonian group action $G \circlearrowright M$ generated by momentum map $\Phi : M \rightarrow \mathfrak{g}^*$ yields dual pair $pt \leftarrow M \xrightarrow{\Phi} \mathfrak{g}^*$ (assume G connected)**

   (a) **Deformation quantization:** $pt \rightsquigarrow \mathbb{C}$**,** $\mathfrak{g}^* \rightsquigarrow C^*(G)$

   (b) **Geometric quantization:** $(M,\omega) \rightsquigarrow [\not{D}^L]_?$

   (c) **Functorial quantization:** $(pt \leftarrow M \rightarrow \mathfrak{g}^*) \rightsquigarrow [\not{D}^L] \in KK_0(\mathbb{C}, C^*(G))$

   $KK_0(\mathbb{C}, C^*(G)) \cong K_0(C^*(G))$ **identifies** $[\not{D}^L] \cong \mu^G_M([\not{D}^L]_{K_0^G(M)})$

3. $(\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) \rightsquigarrow [\not{D} = 0] \in KK_0(C^*(G),\mathbb{C})$**, with** $[C^*(G) \circlearrowright \mathbb{C} \circlearrowleft \mathbb{C}]$

Guillemin-Sternberg-Bott revisited

1. **Composition $\circ$ of dual pairs reproduces symplectic reduction:**

$$(pt \leftarrow M \rightarrow \mathfrak{g}^*) \circ (\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) \cong pt \leftarrow M//G \rightarrow pt$$

General: $(P \rightarrow M \leftarrow Q) \circ (Q \rightarrow N \leftarrow R) = (P \rightarrow (M \times_Q N)/\mathcal{F}_0 \leftarrow R)$

2. **Kasparov product reproduces quantum reduction:**

$$x_{KK_0(\mathbb{C}, C^*(G))} \times_{KK} [\not{D} = 0]_{KK_0(C^*(G),\mathbb{C})} = x^G \in KK_0(\mathbb{C}, \mathbb{C})$$

i.e. map $K_0(C^*(G)) \xrightarrow{x \mapsto x^G} \mathbb{Z}$ **given as product in category** KK_0

3. **Recall:**

$$\begin{aligned} \mathbf{Q}(pt \leftarrow M//G \rightarrow pt) &= \text{index}(\not{D}^{L//G}) \\ \mathbf{Q}(pt \leftarrow M \rightarrow \mathfrak{g}^*) &= \mu_M^G([\not{D}^L]_{K_0^G(M)}) \\ \mathbf{Q}(\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) &= [\not{D} = 0]_{KK_0(C^*(G),\mathbb{C})} \end{aligned}$$

$\Rightarrow$ **Functoriality of quantization map Q gives G-S-B conjecture:**

$$\mathbf{Q}(pt \leftarrow M \rightarrow \mathfrak{g}^*) \circ \mathbf{Q}(\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) = \mathbf{Q}(pt \leftarrow M//G \rightarrow pt)$$

is the same as $\mu_M^G\left(\left[\not{D}^L\right]\right)^G = \text{index}\left(\not{D}^{L//G}\right)$