Functoriality of quantization: a KK-theoretic approach

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Literature


- P. Hochs, Quantisation commutes with reduction for cocompact Hamiltonian group actions, PhD Thesis, Radboud University Nijmegen (2008)

Paths that cross

1. **Symplectic geometry and geometric quantization:**
   - Guillemin–Sternberg (-Dirac) conjecture \([Q, R^G] = 0\)
   - ‘Geometric quantization commutes with symplectic reduction’
   - Reformulation in terms of equivariant index theory (Bott)
   - Defined and proved for *compact* groups and manifolds

2. **Operator algebras and equivariant K-theory:**
   - Baum–Connes conjecture \(\mu_r : K_*(EG) \cong K_*(C^*_r(G))\)
   - Interesting for *noncompact* groups \(G\) (and *proper* actions)

3. **Functoriality of quantization**
   - Can symplectic data ‘neatly’ be mapped into operator data?
   - Are geometric and deformation quantization perhaps related?
The Janus faces of quantization

1. Heisenberg (1925): classical observables $\leadsto$ matrices

2. Schrödinger (1926): classical states $\leadsto$ wave functions

3. von Neumann (1932): unification through Hilbert space
   matrices $\rightarrow$ operators, wave functions $\rightarrow$ vectors

1. Classical observables form Poisson algebra: commutative algebra/$\mathbb{C}$
   and also Lie algebra with Leibniz rule $[fg, h] = f[g, h] + [f, h]g$
   Quantum observables form C*-algebra $\Rightarrow$ first face:
   Deformation quantization: Poisson algebra $\leadsto$ C*-algebra

2. Classical states form Symplectic manifold $(M, \omega) \Rightarrow$ 2nd face:
   Geometric quantization: symplectic manifold $\leadsto$ Hilbert space
   Symplectic form makes $C^\infty(M)$ Poisson algebra w.r.t. $fg(x) = f(x)g(x)$
   and $\{f, g\} = \omega(\xi_f, \xi_g)$, Hamiltonian vector field $\xi_f$: $\omega(\xi_f, \eta) = \eta f$
   Poisson manifold has Poisson algebra structure on $C^\infty(M)$, ibid.
Key examples of quantization

1. **Deformation quantization** (Rieffel)

   Lie group $G$, Lie algebra $\mathfrak{g}$, Poisson mfd $\mathfrak{g}^*$: for $X \in \mathfrak{g}$,
   
   $\hat{X} \in C^\infty(\mathfrak{g}^*)$ defined by $\hat{X}(\theta) = \theta(X)$, $\{\hat{X}, \hat{Y}\} = [X, Y]$

   Quantization of Poisson algebra $C^\infty(\mathfrak{g}^*)$ is $C^*$-algebra $C^*(G)$

2. **Traditional geometric quantization** (Kostant, Souriau)

   compact symplectic manifold $(M, \omega)$ such that $[\omega] \in H^2(M, \mathbb{Z})$
   
   $\Rightarrow$ $\mathbb{C}$-line bundle $L \to M$ plus connection $\nabla^L$ with $F(\nabla^L) = 2\pi i \omega$
   
   $\Rightarrow$ almost complex structure $J$ s.t. $g(\xi, \eta) = \omega(\xi, J\eta)$ is metric
   
   $\Rightarrow$ **Hilbert space** $Q(M, \omega, J) = \{s \in \Gamma(L) \mid \nabla^L_{J\xi - i\xi}s = 0, \xi \in \mathfrak{X}(M)\}$

3. **Postmodern geometric quantization** (Bott)

   $Q_B(M, \omega, J)$ is integer index($\mathcal{D}^L$) := $\dim(\ker(\mathcal{D}^L_+)) - \dim(\ker(\mathcal{D}^L_-))$

   $\mathcal{D}^L$ is Spin$^c$ Dirac operator on $M$ defined by $J$ coupled to $L$
Guillemin-Sternberg conjecture

\( G \triangleleft M \): Lie group action on symplectic \( M \) s.t. momentum map

\( \Phi : M \rightarrow \mathfrak{g}^*, \Phi_X(x) = \langle \Phi(x), X \rangle \) yields \( \mathfrak{g} \)-action: \( \xi^M_X = \xi^\Phi_X (X \in \mathfrak{g}) \)

1. Symplectic reduction: \( M//G = \Phi^{-1}(0)/G \) has symplectic form \( \omega_G \)

2. Geometric quantization: \( Q\left(M//G, \omega_G\right) \) exist if \( Q\left(M, \omega\right) \) exists;
   \( \exists \) line bundle \( (L//G \rightarrow M//G, \nabla^{L//G}) \) with \( F(\nabla^{L//G}) = 2\pi i \omega_G \)

‘Quantization commutes with reduction’: is this the same as

1. Equivariant unitary geometric quantization \( G \triangleleft Q\left(M, \omega\right) \) through Kostant’s formula \( Xs = (\nabla^{L}_\xi^M_X - 2\pi i \Phi_X)s, \ s \in Q\left(M, \omega\right) \subset \Gamma(L) \)

2. ‘Quantum reduction’: \( Q\left(M, \omega\right)//G = Q\left(M, \omega\right)^G \) (Dirac)?

In other words: \( Q\left(M//G, \omega_G\right) \cong Q\left(M, \omega\right)^G \) (as Hilbert spaces)

Proved for \( M \) compact Kähler and \( G \) compact by Guillemin&Sternberg

More general symplectic manifolds require reformulation
Guillemin-Sternberg-Bott conjecture

1. Symplectic reduction: \((M//G, \omega_G)\), same as before

2. Bott’s geometric quantization: \(Q_B(M//G, \omega_G) = \text{index}(\mathcal{D}^{L//G})\)

Bott’s reformulation of G-S conjecture: is this the same as

1. Equivariant geometric quantization \((G&M \text{ compact!})\):
   \[Q_B(M, \omega) = \text{index}_G(\mathcal{D}^L) = [\ker(\mathcal{D}_+^L)] - [\ker(\mathcal{D}_-^L)] \in R(G)\]

2. Quantum reduction: \(Q_B(M, \omega)^G, ([V] - [W])^G = \dim(V^G) - \dim(W^G)\)?

In other words, G-S-B conjecture: \((\text{index}_G(\mathcal{D}^L))^G = \text{index}(\mathcal{D}^{L//G})\)

- Proved by many people in mid 1990s (Meinrenken, . . .)

For noncompact \(G\) and \(M\) need substantial reformulation of G-S-B conjecture, under assumptions: \(G \cap M\) proper and \(M/G\) compact
Noncompact groups and manifolds

Compact ⇔ noncompact dictionary (suggested by Baum-Connes):

- Representation ring $R(G) \rightsquigarrow K_0(C^*(G)) \cong KK_0(\mathbb{C}, C^*(G))$
- $\text{index}_G(\mathcal{D}) \in R(G) \rightsquigarrow \mu^G_M([\mathcal{D}]) \in K_0(C^*(G))$ N.B. $C^*(G)$ not $C^*_r(G)$!

$[\mathcal{D}] \in K^G_0(M) \cong KK^G_0(C_0(M), \mathbb{C})$ equivariant K-homology of $M$

$\mu^G_M : K^G_0(M) \to K_0(C^*(G))$ ‘unreduced’ analytic assembly map (Bunke)

- Quantum reduction $R(G) \to \mathbb{Z} \rightsquigarrow K_0(C^*(G)) \xrightarrow{x \mapsto x^G} K_0(\mathbb{C}) \cong \mathbb{Z}$
  
  induced by map $C^*(G) \to \mathbb{C}$ determined by trivial rep of $G$

$\Rightarrow$ Generalized G-S-B conjecture: $\mu^G_M \left( [\mathcal{D}^L] \right)^G = \text{index} \left( \mathcal{D}^{L//G} \right)$

Epilogue: functorial quantization

- ‘Explains’ generalized Guillemin-Sternberg-Bott conjecture as a special instance of functoriality of quantization
- Unifies the Janus faces of quantization into a functor $Q$

1. Domain of $Q$: Weinstein’s category of (quantizable) ‘dual pairs’
   (a) (integrable) Poisson manifolds as objects
   (b) (regular) symplectic bimodules $[P_1 \leftarrow M \rightarrow P_2] \cong$ as arrows

2. Codomain of $Q$: Kasparov’s category $KK_0$
   (a) $C^*$-algebras as objects
   (b) [Graded Hilbert bimodules $A \odot \mathcal{E} \odot B$ with $\mathcal{D}]_h$ as arrows

3. Hypothetical quantization functor (based on examples only)
   (a) **Deformation quantization:** $P_i \leadsto C^*$-algebra $A_i$
   (b) **Geometric quantization:** $M \leadsto “[\text{Spin}^c \text{ Dirac operator } \mathcal{D}^L]_L”$
   (c) **Functorial quantization:** $P_1 \leftarrow M \rightarrow P_2 \leadsto [\mathcal{D}^L] \in KK_0(A_1, A_2)$
Examples of functorial quantization

1. Symplectic manifold $M$ yields dual pair $pt \leftarrow M \rightarrow pt$
   
   (a) **Deformation quantization:** $pt \rightsquigarrow \mathbb{C}$
   (b) **Geometric quantization:** $(M,\omega) \rightsquigarrow [\mathcal{D}^L]$?
   (c) **Functorial quantization:** $(pt \leftarrow M \rightarrow pt) \rightsquigarrow [\mathcal{D}^L] \in KK_0(\mathbb{C},\mathbb{C})$

   Identification $KK_0(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$ identifies $[\mathcal{D}^L] \cong \text{index}(\mathcal{D}^L)$

2. Hamiltonian group action $G \curvearrowright M$ generated by momentum map $\Phi : M \rightarrow \mathfrak{g}^*$ yields dual pair $pt \leftarrow M \xrightarrow{\Phi} \mathfrak{g}^*$ (assume $G$ connected)
   
   (a) **Deformation quantization:** $pt \rightsquigarrow \mathbb{C}$, $\mathfrak{g}^* \rightsquigarrow C^*(G)$
   (b) **Geometric quantization:** $(M,\omega) \rightsquigarrow [\mathcal{D}^L]$?
   (c) **Functorial quantization:** $(pt \leftarrow M \rightarrow \mathfrak{g}^*) \rightsquigarrow [\mathcal{D}^L] \in KK_0(\mathbb{C},C^*(G))$

   $KK_0(\mathbb{C},C^*(G)) \cong K_0(C^*(G))$ identifies $[\mathcal{D}^L] \cong \mu^G_M([\mathcal{D}^L]_{KK_0(M)})$

3. $(\mathfrak{g}^* \leftarrow 0 \rightarrow pt) \rightsquigarrow [\mathcal{D} = 0] \in KK_0(C^*(G),\mathbb{C})$, with $[C^*(G) \odot \mathbb{C} \odot \mathbb{C}]$
Guillemin-Sternberg-Bott revisited

1. Composition $\circ$ of dual pairs reproduces symplectic reduction:

$$(pt \leftarrow M \rightarrow g^*) \circ (g^* \leftarrow 0 \rightarrow pt) \cong pt \leftarrow M//G \rightarrow pt$$

General: $(P \rightarrow M \leftarrow Q) \circ (Q \rightarrow N \leftarrow R) = (P \rightarrow (M \times_Q N)/F_0 \leftarrow R)$

2. Kasparov product reproduces quantum reduction:

$x_{KK_0(\mathbb{C}, C^*(G))} \times_{KK} [\mathcal{D} = 0]_{KK_0(C^*(G), \mathbb{C})} = x^G \in KK_0(\mathbb{C}, \mathbb{C})$

i.e. map $K_0(C^*(G)) \xrightarrow{x \mapsto x^G} \mathbb{Z}$ given as product in category $KK_0$

3. Recall:

- $Q(pt \leftarrow M//G \rightarrow pt) = \text{index}(\mathcal{D}^{L//G})$
- $Q(pt \leftarrow M \rightarrow g^*) = \mu_M^G([\mathcal{D}^L]_K^G(M))$
- $Q(g^* \leftarrow 0 \rightarrow pt) = [\mathcal{D} = 0]_{KK_0(C^*(G), \mathbb{C})}$

$\Rightarrow$ Functoriality of quantization map $Q$ gives G-S-B conjecture:

$Q(pt \leftarrow M \rightarrow g^*) \circ Q(g^* \leftarrow 0 \rightarrow pt) = Q(pt \leftarrow M//G \rightarrow pt)$

is the same as

$\mu_M^G \left( \left[ \mathcal{D}^L \right] \right)^G = \text{index} \left( \mathcal{D}^{L//G} \right)$