Alex Barnett 6/23/14. ①

use slides.pdf

Tutorial: high-order Nyström quadrature for BIEs (bdry integral eqns).

Goals: intro BIEs, 7 quadratures in 5 key ideas

hope: you read/coded some quadrtut.zip.

Why high-order? digits for hard BVPs, reliability → you can tell if converged.

BVPs eg Ω

$(\Delta + k^2) u = 0$ in Ω (int.) or $\mathbb{R}^d \setminus \Omega$ (ext.) (w/ rad conds.) $d = 2, 3$

$u = f$ on $\partial\Omega$ ← for now.

$k=0$ Laplace

$k>0$ Helmholtz.

Piecewise-const PBEs: Gunnar will tell you why BIE good.

Potential theory: $\Phi(x,y)$ (say (d) 3rc) fund. soln. of PDE: $(\Delta_x + k^2)\Phi(x,y) = \delta(x-y) \quad \forall y \in \mathbb{R}^d$

$x, y \in \mathbb{R}^d$

Single layer pot. $u(x) = \int_{\partial\Omega} \Phi(x,y) \underbrace{\sigma(y)}_{\text{density func.}} ds_y = \mathcal{S}\sigma(x)$

Double " " $u(x) = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y} \sigma(y)\, ds_y = \mathcal{D}\sigma(x)$

} representations: u sat PDE in $\mathbb{R}^d \setminus \partial\Omega$

pics (skip JR)

Parametrize $Z: [0, 2\pi) \to \partial\Omega$

eg $u(x) = \mathcal{S}\sigma(x) = \int_0^{2\pi} \Phi(x, Z(s))\sigma(s) \underbrace{|Z'(s)|}_{\text{length element}} ds$

Fix a quadr rule 2π-periodic funcs: $(s_j, w_j \; j=1\cdots N)$ eg. PTR $j=1 \ldots j=N$, s_j, 0, 2π, w_j ← ask if need to write PTR s_j, w_j?

Gauss-Legendre panels 0, 2π, p nodes.

Nodes $Z(s_j) \in \partial\Omega$ pic

so $u(x) \approx \sum_{j=1}^{N} \Phi(x, Z(s_j)) |Z'(s_j)| \underbrace{\sigma(s_j)}_{\sigma_j}$ good if x far from $\partial\Omega$

BIE: Jump relation: $x \in \partial\Omega$ $u^{\pm}(x) := \lim_{h \to 0^+} u(x \pm h n_x) \overset{\text{say } u = \mathcal{D}\sigma}{=} (D\sigma)(x) \pm \frac{1}{2}\sigma(x)$

x, n_x

$D: C(\partial\Omega) \circlearrowleft$ bdry int. op., same kernel as $\mathcal{D}$.

Say BC was $u^-(x) = \underbrace{(D - I/2)\sigma(x) = f(x)}_{\text{our BIE: solve for } \sigma} \quad \forall x \in \partial\Omega$.

Z. Owens.

RePAram. IE back to $[0,2\pi)$: $\quad \sigma(t) + \int_0^{2\pi} \underbrace{-2\frac{\partial\Phi}{\partial n_{Z(s)}}(Z(t),Z(s))\,|Z'(s)|}_{K(t,s)}\,\sigma(s)\,ds = -2f(t) \quad \forall t\in[0,2\pi)$

$K(t,s)$ (targ. / src) Laplace 2d. $= -2\,\underbrace{\frac{n_{Z(s)}\cdot(Z(t)-Z(s))}{2\pi|Z(t)-Z(s)|^2}\,|Z'(s)|}_{\text{smooth, if } \partial\Omega \text{ is.}}$ (see HW)

Nyström: make hold only @ $t=s_i$ $\quad i=1\cdots N$:

$$\sigma_i + \int_0^{2\pi} K(s_i,s)\sigma(s)\,ds = -2f_i \qquad i=1\cdots N.$$

$$\underbrace{\approx}_{\text{rule}}\ \sum_{j=1}^{N} \underbrace{K(s_i,s_j)w_j}_{A_{ij}\,:\ \text{plain case}}\sigma_j \overset{(1)}{\approx} \sum_{j=1}^{N} A_{ij}\sigma_j$$

then $(I+A)\vec{\sigma} = -2\vec{f}$ $N\times N$ lin sys. $\quad$ solve $\vec{\sigma}$ then use same quadr to eval $\mathcal{D}\sigma$.

culture: Nyström in USA / Galerkin in Europe (BEM).

<u>Singular kernels</u>:

eg $K(t,s) \sim \log|t-s|$, "weak" diagonal singularity. App: $k>0$, $\mathbb{R}^2$, $u = (\mathcal{D} - ik\mathcal{S})\sigma$.

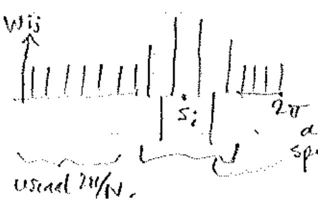
plain rule fails eg $K(s_i,s_i)=\infty$.

seek matrix A st. (1) holds.

(slide): overview, leave up, don't say much.

Idea ①: same nodes, tweak some w_j

a) Kapur-Rokhlin. based on PTR.



a few special w_j for $\overbrace{g(s) + \log|s_i-s|\,h(s)}^{\text{smooth}}$, eg error $O(N^{-6})$

$A_{ij} = K(s_i,s_j)\,w_{ij}$

sparse circulant: $O(N)$ els. $\Rightarrow$ FMM ✓.

easy code; bad constants.

Idea ②: exploit analytic $\quad K(s,t) = \underbrace{\log\left(4\sin^2\frac{t-s}{2}\right)}_{\text{known Fourier series}} K^{(1)}(t,s) + K^{(2)}(t,s)$ ($K^{(1)}$, $K^{(2)}$ smooth)

b) Kress: product quadr.

$$A_{ij} = \underbrace{R_{i-j}}_{\text{dense circulant.}\ \Rightarrow\ \cancel{\text{FMM}}} K^{(1)}(s_i,s_j) + \frac{2\pi}{N}K^{(2)}(s_i,s_j).$$

best for PTR, but analytic effort.

40 mins.

Can use (3) w/ panels? yes:

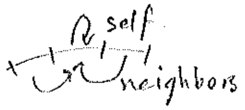
c) Helsing '12 :

Idea (3) : $\sigma(z) \approx \sum_{n=0}^{p-1} c_n z^n$

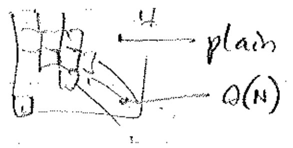
expand in $\mathbb{C}$, not w.r.t. s.

then c_n via Vandermonde solve, $\int_{-1}^{1} z^n \log(y-z)\,dz$ path-indep, recurrence relations.

Fills



, A:



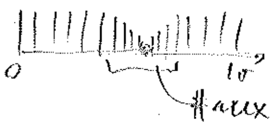
best for panels; tricky.

What if no analytic split?

Idea (4) : in (†) use auxiliary nodes q_ℓ & weights v_ℓ to do $\int_0^{2\pi} K(s_i, s)\,\sigma(s)\,ds$

get via interpolation from local σ_j @ s_j

d) Alpert.



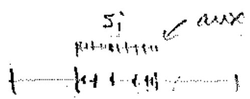
ie local block $A_{ij} = \sum_\ell K(s_i, q_\ell)\, v_\ell\, l_j(q_\ell)$

Lagrange basis for some s_j's.

stable: $v_\ell > 0$

Aux node for panels?

e) Gen. Gaussian. (Kolm-Rokhlin)



precompute sets $\{q_\ell, v_\ell\}$ for each target s_i $i=1\ldots p$. self & nei.

Idea (5): PDE-based close evaluation : $\delta_{ij} + A_{ij} = u^-(Z(s_i))$ for density $\sigma(s) = l_j(s)$

limit of potential on $\partial\Omega$.

f) QBX Klöckner-B-O'Neil-Greengard : d=2or3.

→ slides

50 mins.

Total : 80 mins? ✓

3d: Graham-Sloan, Ganesh.
Wienert.
Bruno-Kunyansky POU.
Bremer
Haroldsen-Meiron
QBX.