On the existence of nonoscillatory phase functions

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This is joint work with Vladimir Rokhlin (Yale University).

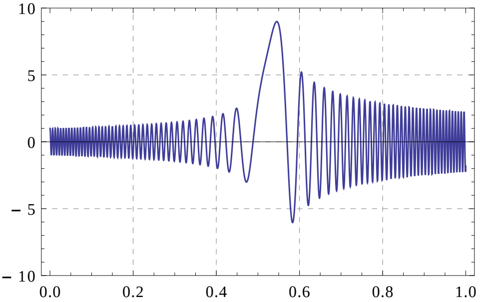
Highly oscillatory ordinary differential equations

I will discuss the efficient representation of solutions of second order linear ordinary differential equations of the form

$$y''(t) + \lambda^2 q(t)y(t) = 0 \quad \text{for all} \quad 0 \leq t \leq 1,$$

where

- $q(t)$ is smooth;
- $q(t)$ is positive on the interval; and
- λ is positive and moderately large.



$q(t) = 1 - \sin(3t)$, $\lambda = 1000$

Solutions are necessarily highly oscillatory.

Representing them accurately with standard methods (e.g., polynomial or trigonometric interpolation) requires a great deal of information, even when $q(t)$ is smooth and can be represented efficiently.

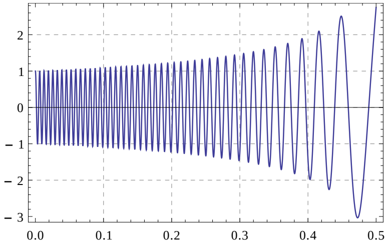
WKB Approximations

The ansatz

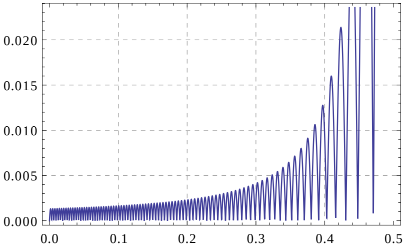
$$y(t) = \exp\left(\sum_{k=0}^{\infty} \lambda^{1-k} b_k(t)\right)$$

gives rise to a sequence of ordinary differential equations. Solving these equations results in asymptotic expansions for two linearly independent solutions.

The functions $b_k(t)$ do not depend on λ and they are roughly as smooth as $q(t)$. Hence, they appear to provide a mechanism for encoding solutions using a quantity of information proportional to that which is required to represent $q(t)$.



First order WKB approximation;
$q(t) = 1 - \sin(3t)$, $\lambda = 1000$.



The absolute error.

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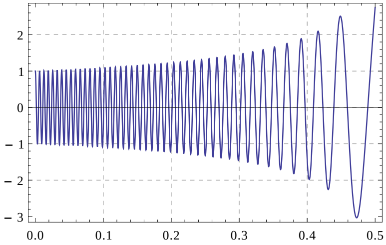
gives rise to a sequence of ordinary differential equations. Solving these equations results in asymptotic expansions for two linearly independent solutions.

The first order WKB approximations are simply

$$q^{-1/4}(t)\cos\left(\lambda \int_0^t \sqrt{q(u)}\, du\right) \quad \text{and} \quad q^{-1/4}(t)\sin\left(\lambda \int_0^t \sqrt{q(u)}\, du\right).$$



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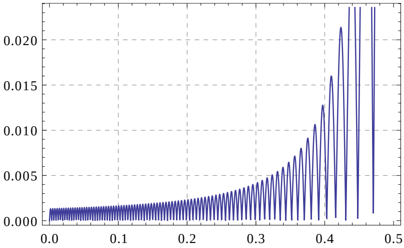


The absolute error.

WKB Approximations need not converge

The m^{th} order WKB approximations is of the form

$$Y_m(t) = \exp\left(\sum_{k=0}^{m} \lambda^{1-k} b_k(t)\right).$$

There exist an exact solution $y(t)$ of the differential equation and a sequence $C_0, C_1, C_2, \cdots$ of constants such that

$$|y(t) - Y_m(t)| \leq C_m \, \lambda^{-m} \quad \text{for all } \; 0 \leq t \leq 1 \; \text{ and all integers } m \geq 0.$$

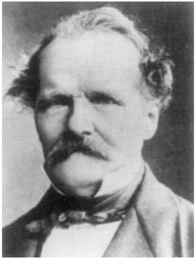
Typically, the approximations $Y_m(t)$ do not converge to $y(t)$. This happens because the sequence of constants C_m grows faster than the sequence λ^{-m} decays.

For a fixed λ, arbitrarily high accuracy cannot necessarily be obtained. Moreover:

$$\text{obtainable accuracy} \sim O\left(\lambda^{-m}\right)$$

for some integer m.

Kummer to the rescue



E.E. Kummer investigated an alternative method for encoding solutions of highly oscillatory ordinary differential equations in the 1840s.

We say that a smooth function $\alpha(t)$ is a phase function for the ordinary differential equation

$$y''(t) + \lambda^2 q(t) y(t) = 0$$

if

$$u(t) = \frac{\cos(\alpha(t))}{|\alpha'(t)|^{1/2}} \quad \text{and} \quad v(t) = \frac{\sin(\alpha(t))}{|\alpha'(t)|^{1/2}}$$

form a basis of solutions.

A function $\alpha(t)$ is a phase function if and only if it satisfies the third order nonlinear ordinary differential equation

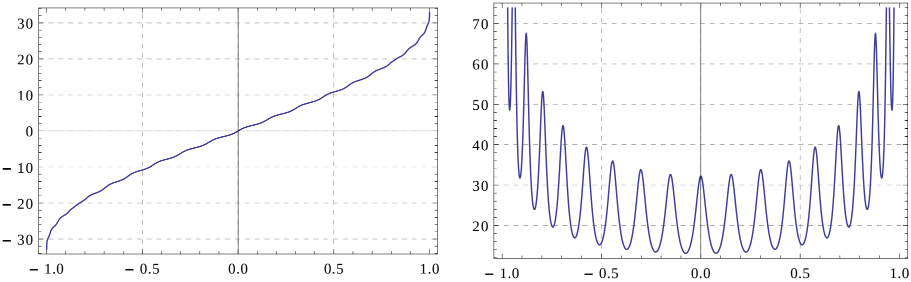
$$\left[\alpha'(t)\right]^2 = \lambda^2 q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2.$$

Phase functions are typically highly oscillatory

Unlike the ordinary differential equations defining WKB approximations, Kummer's equation

$$\left[\alpha'(t)\right]^2 = \lambda^2 q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2$$

contains the wavenumber λ and so we expect its solutions to be highly oscillatory when λ is large.



A phase function for 20th order Legendre polynomials (left) and its derivative (right).

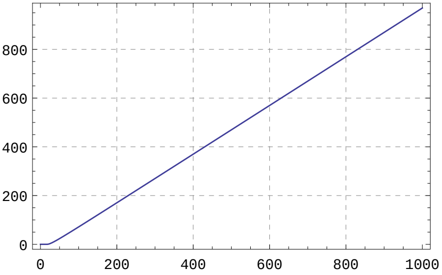
Not all phase functions are oscillatory

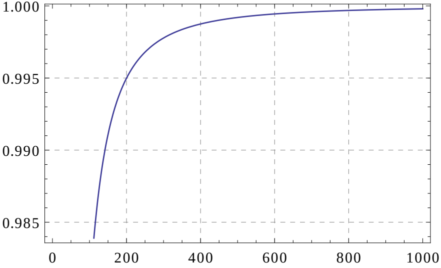
There is a particular well-known phase function $\alpha(z)$ for Bessel's equation which is nonoscillatory.

Asymptotic expansions of it have been standard in mathematical tables for decades.

Its derivative $\alpha'(z)$ is given by the formula

$$\alpha'(z) = \frac{2}{\pi z} \frac{1}{J_\nu^2(z) + Y_\nu^2(z)}.$$





A phase function for 20 order Bessel functions (left) and its derivative (right).

Do nonoscillatory phase functions exist in general?

First key observation

The linearization of the logarithm form of Kummer's equation is the constant coefficient Helmholtz equation in disguise.

An elementary transform converts Kummer's equation (which involves unpleasant quotients) into the more tractable equation

$$r''(t) - \frac{1}{4}\left[r'(t)\right]^2 + 4\lambda^2\left[\exp(r(t)) - q(t)\right] = 0.$$

We call this the **logarithm form of Kummer's equation.**

If we let

$$r(t) = r_0(t) + \delta(t)$$

and neglect terms of order $[\delta(t)]^2$ and higher, then we obtain the ordinary differential equation

$$\delta''(t) - \frac{1}{2}r_0'(t)\delta'(t) + 4\lambda^2\exp(r_0(t))\delta(t) = f(t),$$

where $f(t)$ is nonoscillatory under mild conditions on $r_0(t)$.

Do nonoscillatory phase functions exist in general?

First key observation

The linearization of the logarithm form of Kummer's equation is the constant coefficient Helmholtz equation in disguise.

Under the change of variables

$$x(t) = \int_0^t \exp\left(\frac{r_0(u)}{2}\right)\, du,$$

which is related to the well-known Liouville-Green transform, the equation

$$\delta''(t) - \frac{1}{2} r_0'(t)\delta'(t) + 4\lambda^2 \exp(r_0(t))\delta(t) = f(t),$$

becomes

$$\delta''(x) + 4\lambda^2 \delta(x) = \exp\left(-r(x)\right) f(x),$$

which is simply the constant coefficient Helmholtz equation.

Do nonoscillatory phase functions exist in general?

Second key observation

The constant coefficient Helmholtz equation admits solutions which are essentially nonoscillatory.

If $f(x)$ is a rapidly decaying Schwartz function and we define

$$g(x) = \frac{1}{4\lambda} \int_{-\infty}^{\infty} \sin\left(2\lambda \left|x - y\right|\right) f(y)\ dy,$$

then $g(x)$ is a solution of

$$y''(x) + 4\lambda^2 y(x) = f(x)$$

and

$$\widehat{g}(\xi) = \frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2}.$$

We are interpreting $\widehat{g}(\xi)$ as a tempered distribution defined via a principal value integral.

Do nonoscillatory phase functions exist in general?

Second key observation

The constant coefficient Helmholtz equation admits solutions which are essentially nonoscillatory.

The function

$$\widehat{g}(\xi) = \frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2} = \frac{1}{4\lambda}\left[\frac{\widehat{f}(\xi)}{2\lambda - \xi} + \frac{\widehat{f}(\xi)}{2\lambda + \xi}\right]$$

is singular when $\xi = \pm 2\lambda$, so $g(x)$ has a component which oscillates at frequency 2λ. However, the magnitude of that component is on the order of

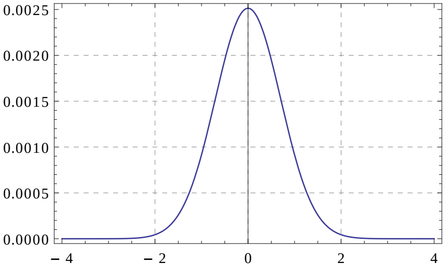
$$\lambda^{-1}\,\widehat{f}(2\lambda).$$

If $\widehat{f}(\xi)$ decays sufficiently rapidly, then this highly oscillatory component will be negligible.

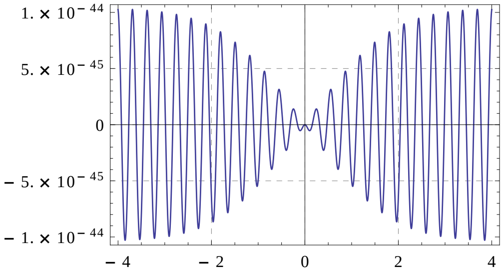
Do nonoscillatory phase functions exist in general?

Second key observation

The constant coefficient Helmholtz equation admits solutions which are essentially nonoscillatory.



This is a solution of $y''(t) + 4\lambda^2 y(t) = f(t)$, where $\lambda = 10$ and $f(t) = \exp(-t^2)$.



Its oscillatory component.

Do nonoscillatory phase functions exist in general?

Key observations:

The linearization of the logarithm form of Kummer's equation is the constant coefficient Helmholtz equation in disguise.

The constant coefficient Helmholtz equation admits solutions which are essentially nonoscillatory.

These observations suggest that we apply Newton's method to the logarithm form of Kummer's equation.

Each iteration consists of solving a linear differential equation which is, after a change of variables, the constant coefficient Helmholtz equation.

We always choose the solution which is essentially nonoscillatory. Moreover, we discard the highly-oscillatory component of negligible magnitude.

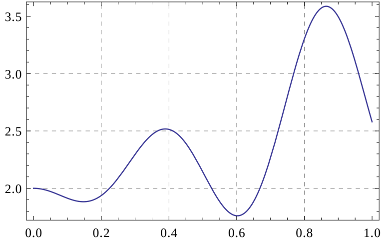
By so doing, we should converge to an approximate solution of Kummer's equation which is nonoscillatory.

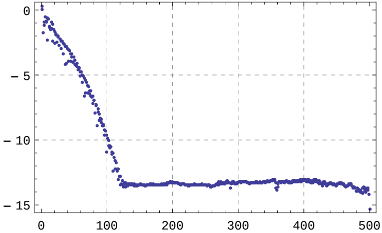
Conclusion:

Solutions of second order linear differential equations can be approximated to high accuracy via nonoscillatory phase functions.

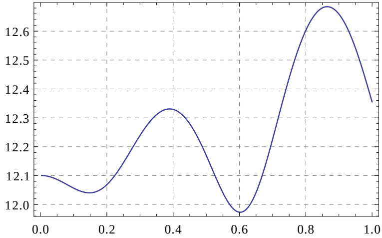
$$q(t) = 2 - \sin(13t)t + t^2; \quad \lambda = 300; \quad \text{iterations} = 6.$$

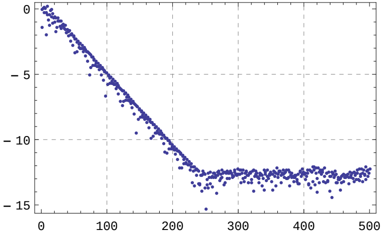
$q(t)$





$r(t)$



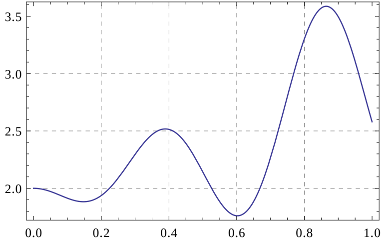


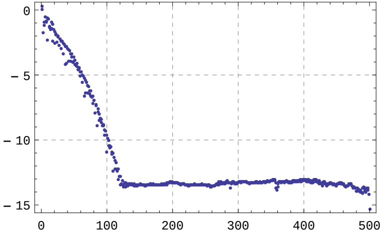
Conclusion:

Solutions of second order linear differential equations can be approximated to high accuracy via nonoscillatory phase functions.

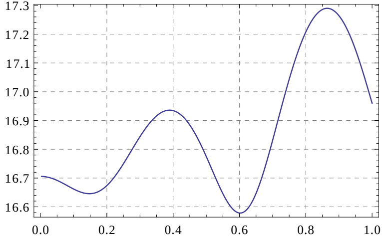
$$q(t) = 2 - \sin(13t)t + t^2; \quad \lambda = 3,000; \quad \text{iterations} = 5.$$

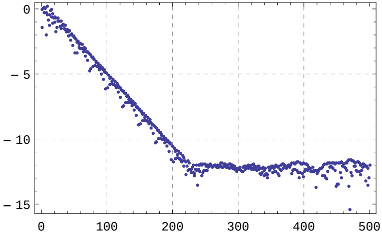
$q(t)$





$r(t)$



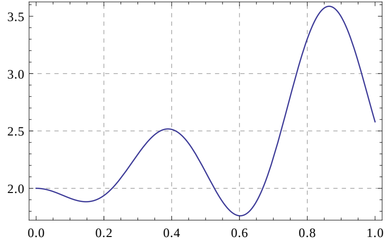


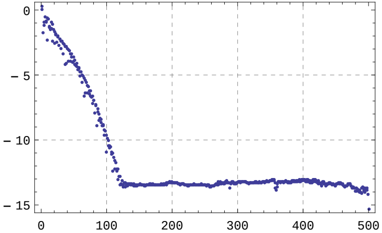
Conclusion:

Solutions of second order linear differential equations can be approximated to high accuracy via nonoscillatory phase functions.

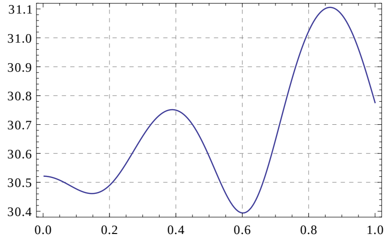
$$q(t) = 2 - \sin(13t)t + t^2; \quad \lambda = 3,000,000; \quad \text{iterations} = 3.$$

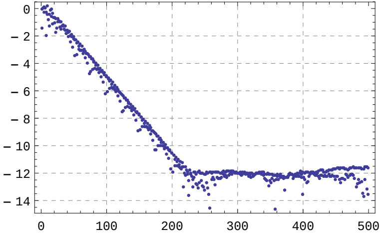
$q(t)$





$r(t)$

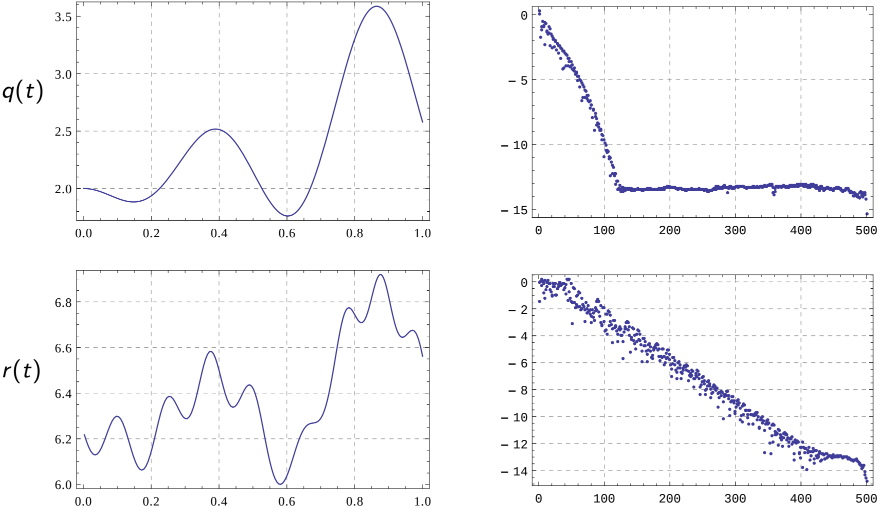




Conclusion:

Solutions of second order linear differential equations can be approximated to high accuracy via nonoscillatory phase functions.

$$q(t) = 2 - \sin(13t)t + t^2; \quad \lambda = 16; \quad \text{iterations} = 12.$$



Reformulation as an integral equation

Through various machinations, we can reformulate the logarithm form of Kummer's equation as the nonlinear integral equation

$$\sigma(x) = S\left[T\left[\sigma\right]\right](x) + p(x),$$

where T is the linear integral operator defined by

$$T\left[f\right](x) = \frac{1}{4\lambda}\int_{-\infty}^{\infty} \sin(2\lambda|x-y|)f(y)\ dy,$$

and S is the nonlinear differential operator defined by

$$S\left[f\right](x) = \frac{\left[f'(x)\right]^2}{4} - 4\lambda^2\left\{\frac{\left[f(x)\right]^2}{2!} + \frac{\left[f(x)\right]^3}{3!} + \cdots\right\}.$$

This equation is obtained with the help of the change of variables

$$x(t) = \int_0^t \sqrt{q(u)}\ du$$

and the function $p(x)$ is related to $q(t)$ through the formula

$$p(x) = 2\{t, x\}.$$

Theorem

Suppose that $\lambda > 0$, $\Gamma > 0$ and $a > 0$ are real numbers such that

$$\lambda > \max\left\{6\Gamma, \frac{6}{a}\right\}.$$

Suppose also that p is a rapidly decaying Schwartz function such that

$$|\widehat{p}(\xi)| \leq \Gamma \exp(-a|\xi|) \quad \text{for all} \quad \xi \in \mathbb{R}.$$

Then there exist a (small) real constant $C > 0$ and functions $\sigma(x)$ and $\widetilde{p}(x)$ such that

$$\|\widetilde{p} - p\|_\infty \leq C\Gamma\lambda \exp(-\lambda a) \quad \text{for all} \quad \xi \in \mathbb{R},$$

$\sigma(x)$ is a solution of the integral equation

$$\sigma(x) = S\left[T\left[\sigma\right]\right](x) + \widetilde{p}(x),$$

and

$$|\widehat{\sigma}(\xi)| \leq 2\Gamma \exp\left(-\left(a - \frac{1}{\lambda}\right)|\xi|\right) \quad \text{for all} \quad \xi \in \mathbb{R}.$$

A few remarks are in order ...

- The solution $\sigma(x)$ of the integral equation

$$\sigma(x) = S\left[T\left[\sigma\right]\right](x) + \widetilde{p}(x)$$

gives rise to a nonoscillatory phase function for a second order linear differential equation

$$y''(t) + \lambda^2 \widetilde{q}(t) y(t) = 0 \quad \text{for all} \quad 0 \leq t \leq 1,$$

where $\widetilde{q}(t) \sim q(t)$.

One can easily bound the difference between solutions of this perturbed equation and the those of the original equation. The relative error is on the order of

$$\lambda \|p - \widetilde{p}\|_\infty \sim \lambda^2 \exp(-a\lambda).$$

- The condition

$$|\widehat{p}(\xi)| \leq \Gamma \exp(-a|\xi|) \quad \text{for all} \quad \xi \in \mathbb{R}$$

is roughly equivalent to requiring that $p(x)$ be bounded and analytic on a strip of radius $a > 0$ around the real axis. The parallels error estimates for Gaussian quadrature are obvious.

A violation of the same space rule

The central difficulty is that the integral equation

$$\sigma(x) = S\left[T\left[\sigma\right]\right] + p(x)$$

breaks the "same space rule." Recall that the operator T is defined by the formula

$$T\left[\sigma\right](x) = \frac{1}{4\lambda}\int_{-\infty}^{\infty} \sin(2\lambda|x-y|)\sigma(y)\ dy.$$

From this definition, we see that in order for $T\left[\sigma\right]$ to exist, at least one of the two following conditions must be met.

- The function σ must decay sufficiently fast at ∞ ($\sigma \in L^1(\mathbb{R})$ suffices);
- $\widehat{\sigma}(\pm 2\lambda) = 0$.

Even if $\sigma(x)$ satisfies both of these conditions, $S\left[T\left[\sigma\right]\right](x)$ will not.

In order for a solution to exist, there must be delicate cancellation involving $p(x)$.

Better living through brutal approximation

In order to overcome this difficulty, we form a "band-limited" version T_b of the linear integral operator T.

The operator T_b is defined for $f \in L^1(\mathbb{R})$ via the formula

$$\widehat{T_b[f]}(\xi) = \frac{\widehat{f}(\xi)b(\xi)}{4\lambda^2 - \xi^2},$$

where $b(\xi)$ is a bump function supported on the interval $[-\lambda, \lambda]$.

T_b is relatively well behaved — for instance, it is bounded as an operator $L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ and it maps of the space of Schwartz functions continuously into itself.

As a consequence, the integral equation

$$\sigma(x) = S[T_b[\sigma]] + p(x)$$

is far more tractable than our original formulation.

A little bit of functional analysis

The contraction mapping principle implies that the sequence of fixed point iterates

$$\begin{aligned}
\sigma_0(x) &= p(x) \\
\sigma_1(x) &= S\left[T_b\left[\sigma_0\right]\right](x) + p(x) \\
&\vdots \\
\sigma_{n+1}(x) &= S\left[T_b\left[\sigma_n\right]\right](x) + p(x) \\
&\vdots
\end{aligned}$$

converges to a solution $\sigma(x)$ of the integral equation

$$\sigma(x) = S\left[\,T_b\left[\sigma\right]\,\right](x) + p(x).$$

Pointwise Fourier estimate

If

$$\left|\widehat{f}(\xi)\right| \leq C\exp(-a|\xi|) \;\text{ for all }\; \xi \in \mathbb{R}$$

then

$$S\,\widehat{[T_b\,[f]]}(\xi) \sim \frac{C^2\exp(-a|\xi|)}{\lambda^4}\left\{\frac{|\xi|^2}{a} + \lambda^2\exp\left(\frac{C}{\lambda^2}|\xi|\right)|\xi|\right\} \quad \text{for all }\; \xi \in \mathbb{R}.$$

This is not too promising at first glance.

But we observe that because of band-limiting, σ_{n+1} only depends on the values of $\widehat{\sigma}_n$ on the finite interval $[-\lambda, \lambda]$, and so

$$\left|\widehat{f}(\xi)\right| \leq C\exp(-a|\xi|) \;\text{ for all }\; |\xi| \leq \lambda$$

implies

$$\begin{aligned} S\,\widehat{[T_b\,[f]]}(\xi) &\sim \frac{C^2\exp(-a|\xi|)}{\lambda^4}\left\{\frac{\lambda^2}{a} + \lambda^2\exp\left(\frac{C}{\lambda^2}\lambda\right)\lambda\right\} \\ &\sim C^2\exp(-a|\xi|)\left\{\frac{1}{\lambda^2 a} + \frac{1}{\lambda}\exp\left(\frac{C}{\lambda}\right)\right\} \end{aligned}$$

for all $|\xi| \leq \lambda$.

Pointwise Fourier estimate

Assuming λ is large relative to C and $\frac{1}{a}$,

$$S\widehat{[T_b[f]]}(\xi) \sim \frac{C^2}{\lambda}\exp(-a|\xi|).$$

From there it is easy to see that

$$|\widehat{\sigma_n}(\xi)| \leq \beta_n \exp(-a|\xi|) \quad \text{for all} \quad |\xi| \leq \lambda,$$

where $\{\beta_n\}$ is defined by

$$\beta_{n+1} = \frac{\beta_n^2}{\lambda} + \beta_0, \quad \beta_0 = \Gamma.$$

As a consequence,

$$|\widehat{\sigma}(\xi)| \leq C\Gamma \exp\left(-\left(a - \frac{1}{\lambda}\right)|\xi|\right) \quad \text{for all} \quad \xi \in \mathbb{R}.$$

Pointwise Fourier estimate

We define $\sigma_b(x)$ via the formula

$$\widehat{\sigma_b}(\xi) = \widehat{\sigma}(\xi) b(\xi),$$

where $b(\xi)$ is the bump function we used to define T_b.

Since the Fourier transform of $\sigma_b(x)$ is 0 at $\pm 2\lambda$, we can plug it into the original integral equation in order to obtain

$$\sigma_b(x) = S\left[T\left[\sigma_b\right]\right](x) + \widetilde{p}(x),$$

where

$$\widetilde{p}(x) = p(x) + \sigma(x) - \sigma_b(x).$$

The estimate

$$\|p - \widetilde{p}\|_\infty \sim \lambda \exp(-a\lambda)$$

follows from the inequality

$$|\widehat{\sigma}(\xi)| \leq 2\Gamma \exp\left(-\left(a - \frac{1}{\lambda}\right)|\xi|\right).$$

Conclusion

- The theorem I've shown you today is far from the best possible result. It can be sharpened in a number of directions — convergence for smaller values of λ, better error estimates, more flexibility in the assumptions on $q(t)$, and so on ...
- There are a number of interesting things to say about the numerical aspects of this subject. They will have to wait for another talk on another day.
- A preprint for the one-dimensional problem is almost ready.
- Generalization of these results to linear elliptic problems in two-dimensions is underway.
- Thank you for your attention!