Reproducing kernel Hilbert $C^*$-module for data analysis *1 *2

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*2 Joint work with Isao Ishikawa, Masahiro Ikeda, Fuyuta Komura, Takeshi Katsura, and Yoshinobu Kawahara
Introduction

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- 2018 Received Master’s degree from Keio University
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Backgrounds / Interests

- Operator theoretic data analysis
- Kernel methods
- Numerical linear algebra
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Kernel methods

\begin{itemize}
  \item Nonlinearity in the original space is transformed into a linear one.
  \item We can compute inner products in RKHS exactly by computers.
\end{itemize}

\begin{itemize}
  \item \textbf{RKHS}\textsuperscript{1} (Infinite dimensional Hilbert sp.)
  \item \textbf{RKHS}\textsuperscript{1} for data analysis
\end{itemize}

\begin{itemize}
  \item \textbf{Feature map} $\phi$
  \item $\phi(x)$ complex-valued function
\end{itemize}

\begin{itemize}
  \item $\mathcal{X}$ (Finite dimensional sp.)
  \item Nonlinear
  \item Linear
  \item Kernel PCA, Kernel SVM
  \item Learning complex-valued functions
\end{itemize}

\begin{itemize}
  \item Advantages of RKHS
\end{itemize}

\textsuperscript{1}Schölkopf and Smola, MIT Press, Cambridge, 2001
Goal: Generalization of data analysis in RKHS to RKHM

\[ x \] sample

Feature map \( \phi \)

\[ \phi(x) \] \( C^* \)-algebra-valued function

\( \mathcal{X} \)
(Structured data sp.)

Nonlinear

\( \mathcal{RKHM} \)
(Infinite dimensional Hilbert \( C^* \)-module)

Linear + \( C^* \)-algebra-valued inner product

Data analysis in RKHM

Advantages of RKHM:

- \( C^* \)-algebra-valued inner products extract information of structures.

We constructed a framework of data analysis with RKHM.

- We can reconstruct existing RKHSs by using RKHMs.
- We have shown fundamental properties for data analysis in RKHMs, similar to RKHSs (e.g. orthogonal projection, representer theorem).
Advantages of RKHM

Algorithms in RKHS

\[ x_1, x_2 : \text{Functional data} \]
\[ x_1, x_2 \in \mathcal{H} \]

\[ \langle x_1, x_2 \rangle_{\mathcal{H}} \in \mathbb{C} \]

\[ c_i = \langle x_1(t_i), x_2(t_i) \rangle_{\mathcal{X}} \in \mathbb{C} \]

\[ \text{Degenerates information along } t \]

Fails to capture continuous behavior (derivatives, total variation, frequency components,...)

Algorithms in RKHM

\[ x_1(t), x_2(t) \in \mathcal{X} \]

\[ \langle x_1(t), x_2(t) \rangle \in \mathbb{C} \]

\[ \langle x_1, x_2 \rangle_{\mathcal{M}} \in \mathcal{A} \]

\[ \text{Capture and control continuous behavior} \]
Review: Reproducing kernel Hilbert space (RKHS)

Let $\mathcal{X}$ be a set. A map $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C}$ is called a positive definite kernel if it satisfies:

1. $k(x, y) = \overline{k(y, x)}$ for $x, y \in \mathcal{X}$ and
2. $\sum_{t,s=1}^{n} c_t k(x_t, x_s) c_s \geq 0$ for $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, $x_1, \ldots, x_n \in \mathcal{X}$.

$\phi(x) := k(\cdot, x)$ ($\phi : \mathcal{X} \rightarrow \mathbb{C}^{\mathcal{X}}$: feature map associated with $k$),

$$\mathcal{H}_{k,0} := \{ \sum_{t=1}^{n} \phi(x_t) c_t \mid n \in \mathbb{N}, \ c_t \in \mathbb{C}, \ x_t \in \mathcal{X} \}. \quad (1)$$

We can define an inner product $\langle \cdot, \cdot \rangle_k : \mathcal{H}_{k,0} \times \mathcal{H}_{k,0} \rightarrow \mathbb{C}$ as

$$\langle \sum_{s=1}^{n} \phi(x_s) c_s, \sum_{t=1}^{l} \phi(y_t) d_t \rangle_k := \sum_{s=1}^{n} \sum_{t=1}^{l} c_s^* k(x_s, y_t) d_t. \quad (2)$$

RKHS $\mathcal{H}_k$: completion of $\mathcal{H}_{k,0}$
Review: Hilbert $C^*$-module

$\mathcal{A}$: $C^*$-algebra, e.g., $\mathcal{A} = B(\mathcal{W}), L^\infty([0, 1])$
(Banach space equipped with a product structure and an involution $* + \alpha$)
$\mathcal{M}$: right $\mathcal{A}$-module ($u \in \mathcal{M}, c \in \mathcal{A} \rightarrow uc \in \mathcal{M}$)

**Definition 1** $\mathcal{A}$-valued inner product

A map $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$ is called an $\mathcal{A}$-valued inner product if it satisfies the following properties for $u, v, w \in \mathcal{M}$ and $c, d \in \mathcal{A}$:

1. $\langle u, vc + wd \rangle = \langle u, v \rangle c + \langle u, w \rangle d$,
2. $\langle v, u \rangle = \langle u, v \rangle^*$,
3. $\langle u, u \rangle \geq 0$ and if $\langle u, u \rangle = 0$ then $u = 0$.

$\rightarrow \mathcal{A}$-valued absolute value $|u| := \langle u, u \rangle^{1/2}$  $\rightarrow$ Norm $\|u\| := \| \langle u, u \rangle \|_A^{1/2}$

**Hilbert $C^*$-module $\mathcal{M}^2$:** complete $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner-product

Short review of reproducing kernel Hilbert $C^*$-module

$\mathcal{A}$: $C^*$-algebra

We can generalize complex-valued notions to operator ($\mathcal{B}(\mathcal{H})$) and function ($L^\infty([0, 1])$)-valued ones. (e.g. eigenvalues, principal components)

RKHS ($\mathcal{H}_k$):
- $\mathbb{C}$-valued positive definite kernel $k$
- $\mathbb{C}$-valued functions
- $\mathbb{C}$-valued inner product

RKHM over $\mathcal{A}$ ($\mathcal{M}_k$):
- $\mathcal{A}$-valued positive definite kernel $k$
- $\mathcal{A}$-valued functions
- $\mathcal{A}$-valued inner product

RKHM for data analysis

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To project a vector onto a finitely generated submodule, we introduce orthonormality\(^3\)

**Definition 2 Orthonormal**

Let \( M \) be a Hilbert \( C^* \)-module.

1. A vector \( q \in M \) is said to be normalized if \( 0 \neq \langle q, q \rangle = \langle q, q \rangle^2 \).
2. Two vectors \( p, q \in M \) are said to be orthogonal if \( \langle p, q \rangle = 0 \).

**Theorem 1 Minimization property**

Let \( A \) be a unital \( C^* \)-algebra and let \( T \) be a finite index set. Let \( V \) be the module spanned by an orthonormal system \( \{q_t\}_{t \in T} \) and let \( P : M \to V \) be the projection operator. For \( w \in M \),

\[
Pw = \arg \min_{v \in V} |w - v|^2
given \text{by (3)}.
\]

Representer theorem in RKHMs

To generalize complex-valued supervised problems to \( \mathcal{A} \)-valued ones, we show a representer theorem.

\[ \mathcal{M}_k: \text{RKHM over } \mathcal{A} \]

**Theorem 2  Representer theorem in RKHMs**

Let \( \mathcal{A} \) be a unital \( C^* \)-algebra, \( x_1, \ldots, x_n \in \mathcal{X} \) and \( a_1, \ldots, a_n \in \mathcal{A} \). Let \( h : \mathcal{X} \times \mathcal{A}^2 \to \mathcal{A}_+ \) be an error function and \( g : \mathcal{A}_+ \to \mathcal{A}_+ \) satisfy \( g(c) < g(d) \) for \( c < d \). If \( \text{Span}_{\mathcal{A}} \{ \phi(x_i) \}_{i=1}^n \) is closed, any \( u \in \mathcal{M}_k \) minimizing \( \sum_{i=1}^n h(x_i, a_i, u(x_i)) + g(|u|_k) \) admits a representation of the form \( \sum_{i=1}^n \phi(x_i)c_i \) for some \( c_1, \ldots, c_n \in \mathcal{A} \).

Key point of the proofs:
For a Hilbert \( C^* \)-module \( \mathcal{M} \) over a unital \( C^* \)-algebra \( \mathcal{A} \) and any finitely generated closed submodule \( \mathcal{V} \) of \( \mathcal{M} \), \( u \in \mathcal{M} \) is decomposed into \( u = u_1 + u_2 \) where \( u_1 \in \mathcal{V} \) and \( u_2 \in \mathcal{V}^\perp \).
Conclusion

- RKHM is a natural generalization of RKHS.

- RKHM enables us to extract continuous behaviors of functional data.

- We showed fundamental properties of RKHM for data analysis.
Ongoing work related to Hilbert $C^*$-modules and dynamical systems*

* Joint work with Isao Ishikawa, Masahiro Ikeda, Suddhasattwa Das, Joanna Slawinska, and Dimitrios Giannakis
Challenges in operator theoretic approaches to analyzing dynamical systems

- Koopman operators (composition operators with respect to dynamical systems) are defined on **infinite-dimensional Hilbert spaces**.
- Koopman operators have **continuous spectra**.
- Continuous spectrum is not described by operators in finite dimensional spaces.
Goal and approach

**Goal:**
Generalize the discrete decomposition in finite-dimensional spaces to that in infinite-dimensional space.

**Approach:**
1. Extend the Koopman operator on a Hilbert space to a Hilbert $C^*$-module.
2. Construct vectors in a Hilbert $C^*$-module using cocycles.
3. Decompose the operator on the Hilbert $C^*$-module using the above vectors.