Boundedness of weighted Koopman operators on a quasi-Banach space and stability of dynamical system

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Self Introduction

Name

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Career

2012–2017　Ph.D (number theory) @Kyoto University, Japan

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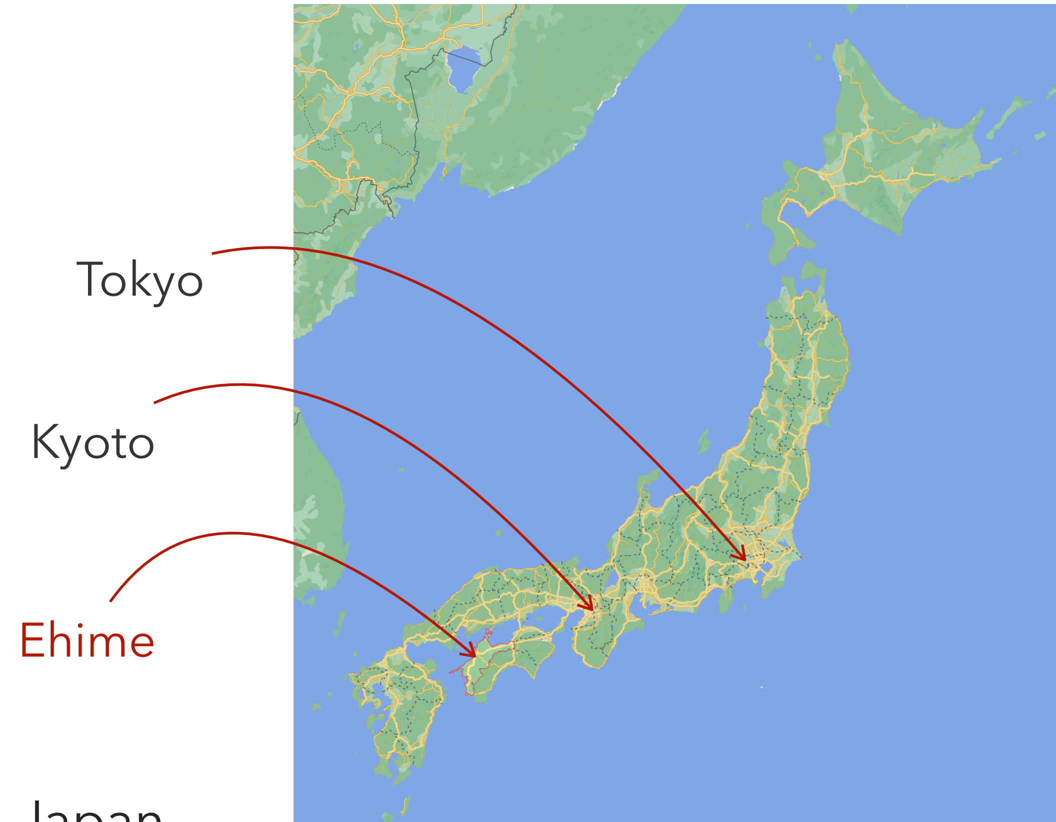
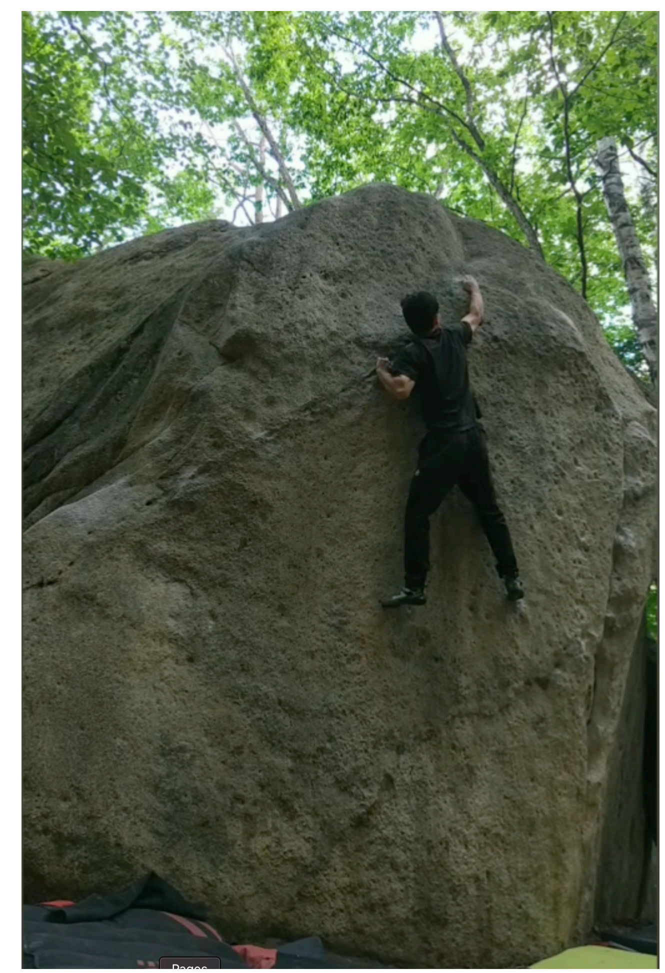
Research Interest

- Theoretical analysis of machine learning
- Koopman operators & data analysis

Hobby

Climbing





Weighted Koopman Operator

Definition

X: set

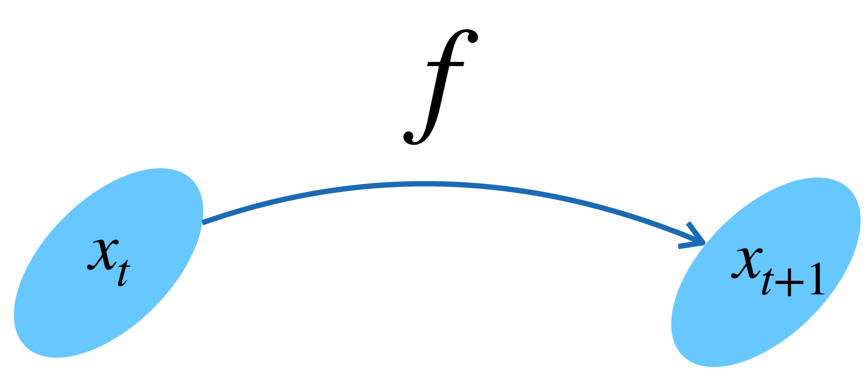
$f: X \to X$: dynamical system

u: $X \to \mathbb{C}$: function

V : function space on X

$uC_f : V \to V$: weighted Koopman operator with weight u is a linear map defined by

$$uC_f[h] := u \cdot (h \circ f)$$



Quasi-Banach Space

Definition (quasi-Banach space)

V: a vector space.

V is a quasi-Banach space if V is a complete topological vector space with respect to the topology determined by a quasi-norm, which is a map $\|\cdot\|_V : V \to \mathbb{R}_{\geq 0}$ such that

- $\|v+w\|_V \leq K(\|v\|_V + \|w\|_V)$ $(K \geq 1)$
- $\|av\| = |a| \cdot \|v\|$ for any $a \in \mathbb{C}$
- $\|v\|_V = 0$ implies $v = 0$

- characterized as a complete Hausdorff topological vector space having a bounded neighborhood of 0.
- We will provides several examples in later slides.

Boundedness

Definition (Boundedness)

V: quasi-Banach space, $A : V \to V$: linear operator

The linear operator A is bounded **if** there exists $L > 0$ such that for any $v \in V$,

$$\|Av\|_V \leq L\|v\|_V$$

- One of basic properties of linear operators.
- Composition operators are recently utilized in data analysis. In data analysis, boundedness is usually needed when we prove a convergence of an estimation algorithm.

Main Question

How much does boundedness restrict the behavior of dynamical system ?

- Boundedness is sometimes essential in application.
- Some existing results show that boundedness sometimes strongly restricts the behavior of dynamical systems.

Setting

X: smooth manifold of dimension d e.g. $\mathbb{R}^d$

$f : X \to X$: smooth map

$\mathscr{E}(X)$: the space of smooth functions on X

(with topology via uniform convergence of any derivatives on compact subsets)

$V \subset \mathscr{E}(X)$: quasi-Banach space

$\iota : V \to \mathscr{E}(X)$: inclusion map

Assume ι is continuous (convergence in $V \Rightarrow$ convergence of all derivatives)

e.g. V = RKHS

Examples of V 1/2

- Reproducing kernel Hilbert space (RKHS)

  $k : X \times X \to \mathbb{C}$: **smooth** positive definite kernel

  k: positive definite iff $\left(k(x_i, x_j)\right)_{i,j=1,\ldots r}$ is a Hermitian positive semi-definite matrix for arbitrarily finite elements $x_1, \ldots, x_r \in X$

  By Moore-Aronsjzan's theorem, $\exists H_k$: Hilbert space in $\mathscr{E}(X)$

  - $k(x, y) = \mathrm{e}^{-(x-y)^2}, X = \mathbb{R}^d$
  - $k(x, y) = \dfrac{\sin(x-y)}{x-y}, X = \mathbb{R}$

Examples of V 2/2

When $p < 1$, V is not Banach but quasi-Banach space.

- Hardy space space

$X = \{z \in \mathbb{C} : |z| < 1\}, p \in (0,\infty]$

$$H^p := \left\{ f : X \to \mathbb{C}\text{: holomorphic : } \sup_{0<r<1} \left(\int_{|z|=r} |f(z)|^p \, dz \right)^{1/p} < \infty \right\}$$

- Fock space

$X = \mathbb{C}^d, p \in (0,\infty]$

$$F^p := \left\{ f : X \to \mathbb{C}\text{: holomorphic : } \left(\int_{|z|=r} |f(z)|^p \mathrm{e}^{-p|z|^2/2} \mathrm{d}A(z) \right)^{1/p} < \infty \right\}$$

- Bergman space, Fock-Sobolev space, e.t.c...

Main Theorem

Theorem 1 (I., arXiv: 2105.04280**)**

Let $p \in X$ be a fixed point of f i.e. $f(p) = p$

Assume uC_f is **bounded**

look at arxiv paper !

If $u(p) \neq 0$ and a **technical condition** for f at p hold, any eigenvalues α of the Jacobian $\mathrm{d}f_p$ of f at p satisfies $|\alpha| \leq 1$.

- If uC_f is bounded, the behavior of f is not chaotic.

The "technical condition" holds for any f and its fixed points if

- V is dense in $\mathscr{E}(X)$ (e.g. universal RKHS),
- $X = \mathbb{R}^d$ and RKHS associated to $k(x, y) = \widehat{w}(x - y)$ for some nonnegative $w \in L^1(\mathbb{R}^d)$,
- $X = \mathbb{C}^d$ and RKHS associated to $k(z, w) = \sum_{n=0}^{\infty} c_n \langle z, w \rangle^n$, or
- X is 1-dimensional complex manifold, f is holomorphic and $\dim V = \infty$

Affineness of Holomorpohic Dynamics (1-dim case)

- An eigenvalue of the Jacobi matrix at a periodic point is an well-studied object in the theory of dynamical system.
- It is well-known that holomorphic dynamics except affine maps on $\mathbb{C}$ are always chaotic.

Corollary 1 (I.)

Assume V is an **infinite-dimensional** and composed of entire functions on $\mathbb{C}$.

Assume uC_f is bounded and one of the following conditions:

- $u = e^w$ for an entire function w,
- f is polynomial,

then $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$

Affineness of Holomorpohic Dynamics (1-dim case)

Theorem 2 (I.)

Assume C_f ($w \equiv 1$ case) is bounded with non-affine $f : \mathbb{C} \to \mathbb{C}$ then

$$V = \{0\} \text{ or } \mathbb{C}1$$

- Affineness holds valid even if $\dim V < \infty$ when $w \equiv 1$

Example

If $V = \mathbb{C}e^z$, $f(z) = z^2 + z$, and $u = e^{-z^2}$, then uC_f induces a bounded linear map on V.

Affineness of Holomorpohic Dynamics (1-dim case)

- There are several works on boundedness of Koopman operators for specific function spaces composed of entire functions, but existing proofs heavily depend on the explicit structure of each function space.

  e.g.

  - Chacon-Chacon-Gimenez '07 by Polya's theorem
  - Hou-Hu-Choi '13 by explicit calculation

- The above result completely determines bounded Koopman operators for 1-dimensional holomorphic dynamics.

Affineness of Holomorphic Dynamics (2-dimensional case)

$\mathscr{G}(V) := \{A \in \mathrm{GL}_d(\mathbb{C}) : C_{A(\cdot)+b}$ is bounded for some $b \in \mathbb{C}^d\}$

Theorem (I.)

Let $d = 2$. Assume

- f is a polynomial automorphism
- $u = \mathrm{e}^{w}$
- $\mathscr{G}(V)$ spans $\mathbb{C}^{2\times 2}$ over $\mathbb{C}$
- the technical condition holds for f and any $p \in \mathbb{C}^2$

If uC_f is **bounded** on V, we have $f(z) = Az + b$ for some $A \in \mathscr{G}(V)$.

- We use a structure theorem of the group of polynomial automorphisms on $\mathbb{C}^2$ by Friedland-Milnor '89

Affineness of Holomorphic Dynamics (general dimensional case)

Conjecture 1

$f : \mathbb{C}^d \to \mathbb{C}^d$: holomorphic map

Assume

- $u = \mathrm{e}^w$
- $\mathcal{G}(V)$ spans $\mathbb{C}^{d \times d}$ over $\mathbb{C}$
- the technical condition holds for f and any $p \in \mathbb{C}^d$

If uC_f is **bounded** on V, we have $f(z) = Az + b$ for some $A \in \mathcal{G}(V)$.

Affineness of Holomorphic Dynamics (general dimensional case)

Conjecture 2

$f : \mathbb{C}^d \to \mathbb{C}^d$: holomorphic map

$\mathscr{G} \subset \mathbb{C}^{d \times d}$: subsemigroup such that $\mathscr{G}$ spans $\mathbb{C}^{d \times d}$

There exists $A \in \mathscr{G}$ such that

$A \circ f$ has saddle or repelling periodic point

- Conjecture 2 implies Conjecture 1
- Conjecture 2 is true in the following cases:
  - $d = 1$
  - $d = 2$ and f is polynomial automorphism

Affineness of Holomorphic Dynamics (general dimensional case)

Conjecture 1 is true in the following cases:

- $d = 1$ and V with $\dim V = \infty$.
- $w \equiv 0$ and $X = \mathbb{R}^d$ and RKHS for $k(x, y) = \hat{\lambda}(x - y)$ for some nonnegative $\lambda \in L^1(\mathbb{R}^d)$ with some condition

  (by Ikeda-I.-Sawano '20)
- $w \equiv 0$ and $X = \mathbb{C}^d$ and RKHS associated to $k(z, w) = \sum_{n=0}^{\infty} c_n \langle z, w \rangle^n$

  (by Doan-Choi-Le '13 and Stochel-Stochel '17)

Conclusion

- Boundedness of weighted Koopman operators makes behavior of dynamical systems tame and non-chaotic, so if we want to treat more chaotic dynamics, we need Koopman theory for unbounded Koopman operators on RKHS (or, we may utilize the boundedness to control the chaotic behavior ?).

- We need to carefully choose function spaces when we need to analyze non-linear dynamical systems (or, can detect or measure linearity of dynamical systems with the affineness results of dynamical systems ?).

- We conjecture the affineness results are also valid in a higher dimensional case.

Our paper is available at arXiv: 2105.04280