Quantum correlations via Operator Systems

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Background

Nonsignalling Games

Figure: Alice

Figure: Bob

$Q \sim$ finite set of questions $\qquad A \sim$ finite set of answers

$$R \subseteq Q^2 \times A^2 \sim \text{finite set of rules}$$

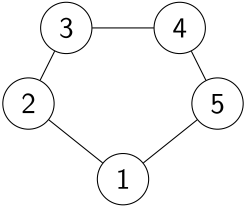
Referee: Picks $x, y \in Q$. $x \mapsto$ Alice, $y \mapsto$ Bob.

Alice: $a \mapsto$ Referee; Bob: $b \mapsto$ Referee.

If $(x, y, a, b) \in R$, then Alice and Bob win. Otherwise, they lose.

Example: 2-coloring game

Consider the 5-cycle:



The two-coloring game: The referee assigns one vertex to Alice and one to Bob, choosing the pair from

$$\{(1,1),(2,2),\ldots,(5,5),(1,2),(2,3),\ldots,(5,1)\}$$

Receive same vertex: Alice and Bob win if they return same color.
Receive adjacent vertices: Alice and Bob win if they return different colors.
What strategy maximizes their chances of winning?

Correlations

A **correlation** is a tuple

$$p = \{p(a, b|x, y)\}_{a,b\in A,x,y\in Q}$$

satisfying

$$p(a, b|x, y) \geq 0 \quad \text{and} \quad \sum_{a,b\in A} p(a, b|x, y) = 1.$$

A correlation p is **nonsignalling** if the **marginal densities** given by

$$p_A(a|x) := \sum_{c\in A} p(a, c|x, w) \quad \text{and} \quad p_B(b|y) := \sum_{c\in A} p(c, b|z, y)$$

are well-defined.

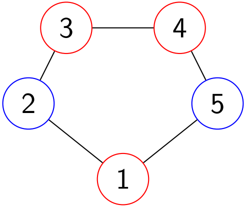
Deterministic strategy

A correlation is **deterministic** if $p_A(a|x), p_B(b|y) \in \{0, 1\}$.

Alice has $f : Q \to A$; Bob has $g : Q \to A$. If $x \mapsto$ Alice and $y \mapsto$ Bob, Alice: $f(x) \mapsto$ Referee and Bob: $g(y) \mapsto$ Referee.

Example

In the two coloring game, an optimal deterministic strategy is: Alice and Bob agree on a coloring which is "almost" a two-coloring. Chance of winning: 90%



Local strategy

A correlation is **local** if it has the form

$$p = \sum_{i=1}^{k} t_i p_i$$

where $t_i \geq 0, \sum t_i = 1$, and $p_1, \ldots, p_k$ are deterministic.

Example

In the two coloring game, an optimal local strategy is deterministic: Chance of winning: 90%

Quantum strategy

Quantum principles:

- physical systems $\cong H$,
- state of physical system $\cong \phi \in H$,
- measurements $\cong \{P_a\}_{a=1}^m$ with $P_a^2 = P_a$ and $\sum_a P_a = I$.
- Observe a with probability $\langle P_a\phi, \phi\rangle = \|P_a\phi\|^2$.

A correlation p is called a **quantum correlation** if there exist finite dimensional Hilbert spaces H_A, H_B, a unit vector $\phi \in H_A \otimes H_B$, and measurements $\{E_{x,a}\} \subseteq B(H_A)$ and $\{F_{y,b}\} \subseteq B(H_B)$ such that

$$p(a, b|x, y) = \langle E_{x,a} \otimes F_{y,b}\phi, \phi\rangle.$$

Example (Cleve-Hoyer-Toner-Watrous)

In the 5-cycle 2-coloring game, the optimal quantum strategy allows Alice and Bob win with probability

$$\frac{1}{2} + \frac{1}{2}\cos(\pi/10) \cong 97.6\%$$

Entanglement and quantum correlations

A state $\phi \in H_A \otimes H_B$ has the form $\sum \lambda_i \psi_i \otimes \rho_i$ where $\psi_i \in H_A$ and $\rho_i \in H_B$.

A state ϕ is called **separable** if it can be written as $\phi = \phi_A \otimes \phi_B$ with $\phi_A \in H_A$ and $\phi_B \in H_B$. If not, ϕ is called **entangled**.

Theorem (Bell)

If ϕ is separable, then the quantum correlation $p(a, b|x, y) = \langle E_{x,a} \otimes F_{y,b}\phi, \phi\rangle$ is a local correlation.

So if a quantum strategy outperforms a local strategy, it must require an entangled state...

Quantum commuting correlations

A correlation p is called a **quantum commuting correlation** if there exists a Hilbert space H, a unit vector $\phi \in H$, and measurements $\{E_{x,a}, F_{y,b}\} \subseteq B(H)$ satisfying

$$E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$$

such that

$$p(a, b|x, y) = \langle E_{x,a}F_{y,b}\phi, \phi\rangle.$$

Example

In the 5-cycle 2-coloring game, the optimal quantum commuting strategy allows Alice and Bob win with probability 97.6%... same as quantum.

Quantum correlations in theory

For $|Q| = n, |A| = m$, let $C_*(n, m)$ be all correlations of type $*$.

We have $C_{loc}(n, m) \subseteq C_q(n, m) \subseteq C_{qc}(n, m) \subseteq C_{ns}(n, m) \subseteq \mathbb{R}^{n^2 m^2}$.

Bell (1960s): $C_{loc}(n, m) \neq C_q(n, m); n, m \geq 2$.

Fritz, Junge et. al. (2011): A positive solution to *Connes' embedding problem* (ca. 1970) implies the equality $\overline{C_q(n, m)} = C_{qc}(n, m)$ for all n, m.

Slofstra (2019): $C_q(n, m) \neq C_{qc}(n, m)$; $n = 235, m = 8$.

Ji-Natarajan-Vidick-Wright-Yuen (Preprint, 2020): $\overline{C_q(n, m)} \neq C_{qc}(n, m); n, m \cong 10^{20}$.

Generating correlations from abstract data

Given a Hilbert space H,

- a **C*-algebra** is a self-adjoint closed unital subalgebra of $B(H)$.
- an **operator system** is a self-adjoint unital subspace of $B(H)$.

C*-algebras and operator systems also have "abstract" definitions.

- An **abstract C*-algebra** is a normed $*$-algebra satisfying $\|x\|^2 = \|x^*x\|$.
- An **abstract operator system** is a $*$-vector space V together with a sequence of positive cones $C_n \subseteq M_n(V)_h$ and unit e.

Given an abstract C*-algebra (abstract operator system) there exists a Hilbert space H and an isomorphic C*-subalgebra (operator subsystem) of $B(H)$.

Correlations from C*-algebras

A **state** on a C*-algebra (or operator system) is a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(I) = 1$ and $\phi(x) \geq 0$ whenever $x \geq 0$.

Theorem (Folklore)

A correlation p is quantum commuting if and only if there exists a C-algebra $\mathcal{A}$, projection-valued measures $\{E_{x,a}\}, \{F_{y,b}\} \subseteq \mathcal{A}$ with $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ and a state ϕ such that $p(a,b|x,y) = \phi(E_{x,a}F_{y,b})$. Moreover,*

- *$p \in C_q(n,m)$ if and only if the statement holds for a finite-dimensional $\mathcal{A}$.*
- *$p \in C_{loc}(n,m)$ if and only if the statement holds for a commutative $\mathcal{A}$.*

Correlations from operator systems?

Assume $p \in C_{qc}(n, m)$ with $p(a, b|x, y) = \langle E_{x,a} F_{y,b} \phi, \phi \rangle$. Then:

$$\mathcal{V} = \mathrm{span}\{E_{x,a} F_{y,b}\}$$

is an operator system and $E_{x,a} F_{y,b} \mapsto \langle E_{x,a} F_{y,b} \phi, \phi \rangle$ is a state on $\mathcal{V}$.

So correlations only require a finite dimensional operator system and a state.

We can abstractly characterize operator systems, but what about operator systems of the form

$$\mathcal{V} = \mathrm{span}\{E_{x,a} F_{y,b}\}?$$

Compressions of operator systems

Proposition (Araiza-R.)

Let $p \in \mathcal{V} \subseteq B(H)$, $x \in \mathcal{V}$ with $x = x^*$. Then $pxp \geq 0$ if and only if for every $\epsilon > 0$ there exists a $t > 0$ such that

$$x + \epsilon p + t(I - p) \in \mathcal{V}_+.$$

Thus if $p \in \mathcal{V}$ is a projection, we can detect when $pxp \geq 0$ using only the data of the operator system $(\mathcal{V}, \{C_n\}, e)$.

Characterization of projections

Assume $p \in \mathcal{V} \subseteq B(H)$ and p is a projection. Let $q = I - p$. Then we may decompose each $x \in \mathcal{V} \subseteq B(H) = B(pH \oplus qH)$ as

$$x = \begin{pmatrix} pxp & pxq \\ qxp & qxq \end{pmatrix}.$$

Consider the compression of

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \text{ by } p \oplus q, \text{ i.e. } \begin{pmatrix} \begin{pmatrix} pxp & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & pxq \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ qxp & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & qxq \end{pmatrix} \end{pmatrix}.$$

Observe that $x \geq 0$ if and only if $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ has positive compression by $p \oplus q$.

Characterization of projections

Definition

We call a positive contraction p in an abstract operator system $\mathcal{V}$ an **abstract projection** if the set of $x = x^* \in M_n(\mathcal{V})$ satisfying for every $\epsilon > 0$ there exists $t > 0$ such that

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} + \epsilon I_n \otimes (p \oplus q) + t I_n \otimes (q \oplus p) \geq 0$$

coincides with the positive cone of $M_n(\mathcal{V})$.

Theorem (Araiza, R.)

A positive contraction p in an operator system $(\mathcal{V}, \{C_n\}, e)$ is an abstract projection if and only if there exists a unital complete order embedding $\pi : \mathcal{V} \to B(H)$ such that $\pi(p)$ is a projection.

The set of abstract projections

The theorem allows us to build $\pi : \mathcal{V} \to B(H)$ mapping a single abstract projection p to an honest projection $\pi(p)$. What if there are many abstract projections?

Theorem (Araiza-R.)

Let p be an abstract projection in an operator system $\mathcal{V}$. Then p is a projection in $C_e^*(\mathcal{V})$.

Thus, if $p_1, p_2, \ldots, p_N \in \mathcal{V}$ are all abstract projections, then $p_1, p_2, \ldots, p_N$ are projections in $C_e^*(\mathcal{V})$.

Quantum commuting operator systems

Definition

A **quantum commuting operator system** is a finite dimensional operator system with unit e spanned by positive contractions $\{Q(a,b|x,y) : a,b \in [m], x,y \in [n]\}$ such that

- For each $x,y \in [n]$, $\sum_{a,b\in[m]} Q(a,b|x,y) = e$
- For each $x,y \in [n]$ and $a,b \in [m]$, the vectors

  $$E(a|x) := \sum_{c\in[m]} Q(a,c|x,w) \quad \text{and} \quad F(b|y) := \sum_{c\in[m]} Q(c,b|z,y)$$

  are well-defined
- Each generator $Q(a,b|x,y)$ is an abstract projection.

Quantum correlations and operator systems

Theorem (Araiza, R.)

A correlation p is quantum commuting if and only if there exists a quantum commuting operator system $\mathcal{V} = span\{Q(a, b|x, y)\}$ and a state $\phi : \mathcal{V} \to \mathbb{C}$ such that

$$p(a, b|x, y) = \phi(Q(a, b|x, y)).$$

Proof elements:

- The linear relations between $\{Q(a, b|x, y)\}$ ensure p is nonsignalling correlation.
- Each $Q(a, b|x, y)$ is a projection in $C_e^*(\mathcal{V})$.
- The relations $Q(a, b|x, y) = E(a|x)F(y|b) = F(y|b)E(a|x)$ are forced.

An operator system $\mathcal{V}$ is k-**minimal** if it is isomorphic to a subsystem of $\oplus_{i \in I} M_k$.

Proposition (Araiza-R.-Tomforde)

$\mathcal{V}$ is k-minimal if and only if

$$C_n = \{x \in M_n(\mathcal{V})_h : \alpha^* x \alpha \in C_k \text{ for all } \alpha \in M_{n,k}\}$$

for each $n > k$.

An operator system is **matricial** if it is k-minimal for some $k \in \mathbb{N}$.

Quantum correlations and operator systems

Theorem (Araiza-R.-Tomforde)

A correlation p is quantum if and only if there exists a matricial quantum commuting operator system $\mathcal{V} = span\{Q(a,b|x,y)\}$ and a state $\phi : \mathcal{V} \to \mathbb{C}$ such that

$$p(a,b|x,y) = \phi(Q(a,b|x,y)).$$

Proof elements:

- The previous theorem tells us p is quantum commuting.
- Each $Q(a,b|x,y)$ is a projection in $C_e^*(\mathcal{V})$, and $C_e^*(\mathcal{V})$ is a C*-subalgebra of $\oplus M_k$.
- Using Caratheodory's theorem, p can be written as a finite convex combination of quantum correlations.

Correlations from operator systems

Corollary

Suppose $\mathcal{V} = span\{Q(a, b|x, y)\}$ *is an operator system and* $\phi : \mathcal{V} \to \mathbb{C}$ *is a state. Let* $p(a, b|x, y) = \phi(Q(a, b|x, y))$.

1. *If* $\mathcal{V}$ *quantum commuting, then* $p \in C_{qc}(n, k)$.
2. *If* $\mathcal{V}$ *is quantum commuting and* d*-minimal for some* d*, then* $p \in C_q(n, k)$.
3. *If* $\mathcal{V}$ *is quantum commtuing and 1-minimal, then* $p \in C_{loc}(n, k)$.

Problem: Construct a quantum commuting operator system $\mathcal{V} = \text{span}\{Q(a, b|x, y)\}$ which cannot be approximated by a d-minimal quantum commuting operator system for any d.

Thanks!

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