Independence of Weyl-Heisenberg coherent states: the HRT Conjecture

Abstract: In 1932, J. Von Neumann proposed that the coherent states

$$\mathcal{G}(g,1,1) = \{g_{n,k}(t) := e^{2\pi ikt}g(t-n) : k,n \in \mathbb{Z}\},$$

generated by the Gaussian $g(t) = e^{-\pi t^2}$, span a dense subspace of $L^2(\mathbb{R})$. A decade later, D. Gabor independently formulated a similar idea in the context of communication theory. These foundational observations gave rise to modern time–frequency analysis, which centers on coherent state systems of the form

$$\mathcal{G}(g,\Lambda) = \{g_\lambda(t) := e^{2\pi i\lambda_2 t}g(t-\lambda_1) : \lambda = (\lambda_1,\lambda_2) \in \Lambda\},$$

where $g \in L^2(\mathbb{R})$ and $\Lambda \subset \mathbb{R}^2$ is a discrete set.

Within this framework, C. Heil, J. Ramanathan, and P. Topiwala (1996) conjectured that for any nonzero square-integrable function g and any finite subset $\Lambda = \{(a_k,b_k)\}_{k=1}^N \subset \mathbb{R}^2$, the corresponding (finite) Weyl-Heisenberg system

$$\mathcal{G}(g,\Lambda) = e^{2\pi ib_k\cdot}g(\cdot - a_k)_{k=1}^N$$

is linearly independent. This statement, now known as the HRT Conjecture, remains a central open problem in time–frequency analysis despite sustained efforts over nearly three decades.

In this talk, I will first provide a historical overview of the conjecture, and then present recent progress on cases where the discrete set Λ is at most five.