CONNECTING FRAME THEORY, HARMONIC ANALYSIS, AND OPERATOR THEORY VIA DYNAMICAL SAMPLING

VICTOR BAILEY

A frame for a separable Hilbert space, H, is a countable collection of vectors in H that satisfy the following inequality for every $f \in H$,

$$C_1\|f\|^2 \leq \sum_{n\in I} |\langle f, f_n\rangle|^2 \leq C_2\|f\|^2$$

where $0 < C_1 \leq C_2$ are fixed constants. A dynamical frame is a frame for H obtained by iterating a bounded operator(s) on a fixed vector(s). A central problem in dynamical sampling (i.e. dynamical frame theory) is the following: Given an operator $T \in B(H)$ and a vector $\varphi \in H$, what are necessary and sufficient conditions so that $\{T^n\varphi\}_{n\in\mathbb{Z}^+}$ is a frame for H? When H is infinite-dimensional, a result due to Christensen et. al. shows that the only operators which can generate a frame for H in this way, must be similar to a compression of the shift operator on the Hardy space on the unit disk, $H^2(\mathbb{D})$, to some coinvariant subspace of the shift operator in $H^2(\mathbb{D})$. This result and several other subsequent results regarding dynamical frames have been obtained by applying tools from operator theory. To date there are a number of results connecting frame theory to the theory of model spaces, analytic Hilbert modules, C_0-semigroups and Hille-Phillps functional calculus, etc. as tools from each of these areas have been extensively used to solve problems posed in dynamical sampling.

On the other hand, there are important questions in operator theory that can be approached naturally from the lens of dynamical sampling. It is well known that Beurling's Invariant Subspace Theorem, which characterizes the structure of all the shift-invariant subspaces in $H^2(\mathbb{D})$, does not extend to $H^2(\mathbb{D}^2)$. The study of the structure of shift-invariant subspaces of $H^2(\mathbb{D}^2)$ has been the subject of extensive work over the past several decades and to date there is still no complete characterization of their structure. Dynamical sampling provides a new way to approach this classification problem. Associated with every frame of the form $\{T_1^iT_2^j\varphi\}_{i,j\in\mathbb{Z}^+}$ (where $T_1, T_2 \in B(H)$ commute) is a corresponding shift-invariant subspace of $H^2(\mathbb{D}^2)$ whose structure and properties depend on the frame. Due to this, we can consider this problem in operator theory from the viewpoint of frames by iterations of bounded commuting operator pairs. Additionally, given an bounded operator $T \in B(H)$, finding the conditions needed so that the operator admits a cyclic vector, that is a vector $\varphi \in H$ such that $\overline{span}\{T^n\varphi\}_{n\in\mathbb{Z}^+} = H$,

Date: 8 October 2026.

is a problem that is often posed in operator theory. Frame vectors for an operator are not only cyclic, but they also provide basis-like expansions for any $f \in H$ via the frame $\{T^n\varphi\}_{n\in\mathbb{Z}^+}$. Any question regarding finding cyclic vectors for an operator naturally gives way to a question of finding frame vectors for the operator and there are various works which provide both conditions for an operator to admit a frame vector and a means to distinguish between frame vectors for an operator and those that are merely cyclic.

In this talk, we will discuss the complementary nature of these research areas within the framework of dynamical sampling. We will also discuss recent results from joint work with Eva Gallardo-Gutierrez and Jonathan Partington regarding continuous frames generated by a C_0-semigroup and recent results from joint work with Rongwei Yang and Kelly Bickel regarding the exact form properties of the inner function defining the Izuchi-Ohno type submodule in $H^2(\mathbb{D}^2)$.

Morehouse College
Email address: victor.bailey@morehouse.edu