INTRINSIC DENSITY, ASYMPTOTIC COMPUTABILITY, AND STOCHASTICITY

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Abstract

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There are many computational problems which are generally "easy" to solve but have certain rare examples which are much more difficult to solve. One approach to studying these problems is to ignore the difficult edge cases. Asymptotic computability is one of the formal tools that uses this approach to study these problems. Asymptotically computable sets can be thought of as almost computable sets, however every set is computationally equivalent to an almost computable set. Intrinsic density was introduced as a way to get around this unsettling fact, and which will be our main focus.

Of particular interest for the first half of this dissertation are the intrinsically small sets, the sets of intrinsic density 0. While the bulk of the existing work concerning intrinsic density is focused on these sets, there are still many questions left unanswered. The first half of this dissertation shall endeavor to answer some of these questions. We shall prove some useful closure properties for the intrinsically small sets and apply them to prove separations for the intrinsic variants of asymptotic computability. We shall also completely separate hyperimmunity and intrinsic smallness in the Turing degrees and resolve some open questions regarding the relativization of intrinsic density.

For the second half of this dissertation, we shall turn our attention to the study of intermediate intrinsic density. We shall develop a calculus using noncomputable coding operations to construct examples of sets with intermediate intrinsic density. For almost all $r \in (0, 1)$, this construction will yield the first known example of a set with intrinsic density r which cannot compute a set random with respect to the r-Bernoulli measure. Motivated by the fact that intrinsic density coincides with the notion of injection stochasticity, we shall apply these techniques to study the structure of the more well-known notion of MWC-stochasticity.

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CHAPTER 1

INTRODUCTION

1.1 Background

We shall use relatively standard notation from computability theory. Appendix A contains a review of all notation and conventions we rely upon.

1.1.1 Asymptotic Computability

A noteworthy phenomenon in the world of computing is that of problems which are generally "easy" to compute but have very difficult worst case instances. This gave rise to the notion of *generic computability*, studied by Kapovich, Myasnikov, Schupp, and Shpilrain [13] in the context of computing the word problems of finitely generated groups. This notion asserts that a set is computable outside of a "small" error set where the algorithm does not answer. The notion of smallness here is that of having asymptotic density 0:

Definition 1.1. The partial density of $A \subseteq \omega$ at n is

$$\rho_n(A) = \frac{|A \upharpoonright n|}{n}.$$

That is, it is the ratio of the number of things less than n that are in A to what could be in A. The upper (asymptotic) density of A is

$$\overline{\rho}(A) = \limsup_{n\to\infty} \rho_n(A)$$

and the lower (asymptotic) density of A is

$$\underline{\rho}(A) = \liminf_{n\to\infty} \rho_n(A).$$

If $\overline{\rho}(A) = \underline{\rho}(A)$, we call this limit the (asymptotic) density of A and denote it by $\rho(A)$.

It is important to note that $\underline{\rho}(A) = 1$ implies $\rho(A) = 1$ and $\overline{\rho}(A) = 0$ implies $\rho(A) = 0$.

Informally, a set is asymptotically computable if there is a Turing machine that converges and agrees with the set's characteristic function on a set of asymptotic density 1. The error set is the set of asymptotic density 0 where the Turing machine does not do this. However, there are multiple

ways that this could happen. Requiring the Turing machine to behave in certain ways yields a total of four separate notions of asymptotic computability.

Definition 1.2. A set A is generically computable if there is a partial computable function φ_e such that $\underline{\rho}(W_e) = 1$ and if $\varphi_e(n)\downarrow$, then $\varphi_e(n) = A(n)$. The function φ_e is called a generic description of A.

We think of generically computable sets as being computable "almost everywhere," i.e. there is an algorithm that correctly answers questions on a set of density 1, but does not answer on the error set. Here the error set is the set of n on which the description diverges.

Definition 1.3. A set A is coarsely computable if there is a total computable function φ_e such that $\underline{\rho}(\{n : \varphi_e(n) = A(n)\}) = 1$. The function φ_e is called a coarse description of A.

For coarse computability, the description is forced to answer every question, but is allowed to give the incorrect answer on the error set. That is, the error set is the set of numbers on which the description and the set disagree.

Definition 1.4. A set A is densely computable if there is a partial computable function φ_e such that $\underline{\rho}(\{n : \varphi_e(n)\downarrow = A(n)\}) = 1$. The function φ_e is called a dense description of A.

For dense computability, the description can both answer questions incorrectly and not answer them on the error set. More specifically, the error set consists of both the places where the description diverges and those where it converges but disagrees with the characteristic function.

Definition 1.5. A set A is effectively densely computable if there is a total computable function $\varphi_e : \omega \to \{0, 1, \square\}$ such that $\underline{\rho}(\varphi_e^{-1}(\{0, 1\})) = 1$ and $\varphi_e(n) \in \{0, 1\}$ implies $\varphi_e(n) = A(n)$.

Effective dense computability, on the other hand, must answer correctly everywhere it answers and must converge on all n. However, it may refuse to answer, which is represented by converging to $\square$. In particular, the error set, which is the inverse image of $\square$ under the description, must be computable.

Note that there are some obvious implications between these notions. Effective dense computability implies both coarse computability and generic computability, and both of these imply dense computability. For an overview of the history of these notions, refer to the first section of [5].

One potentially unsettling feature of all four notions of asymptotic computability is that they depend heavily on the way in which information is coded. In fact, Jockusch and Schupp [12] give a simple argument that can show every Turing degree contains a set which is effectively densely

computable by "hiding" an entire set of any degree on a small computable set such as the set of factorials. (As the other three notions are implied by effective dense computability, the same is automatically true for every notion of asymptotic computability.)

Proposition 1.6. Let $X \subseteq \omega$. Then there is $A \equiv_T X$ which is effectively densely computable.

Proof. Given X, let $A = \{n! : n \in X\}$. Then A is clearly Turing equivalent to X, and the function

$$f(n) = \begin{cases} \square & \text{if } n = k! \\ 0 & \text{otherwise} \end{cases}$$

witnesses that A is effectively densely computable. □

This justifies the idea that these notions of being "almost" computable are heavily dependent upon how the set is coded: computably re-arranging the elements of a set can break the property of being "almost computable," and any computational problem is equivalent to an "almost" computable problem.

1.1.2 Intrinsic Density

To combat this instability, Astor [4] introduced the notion of *intrinsic density*, a strengthening of asymptotic density.

Definition 1.7. The absolute upper density of $A \subseteq \omega$ is

$$\overline{P}(A) = \sup\{\overline{\rho}(\pi(A)) : \pi \text{ a computable permutation}\}$$

and the absolute lower density of A is

$$\underline{P}(A) = \inf\{\underline{\rho}(\pi(A)) : \pi \text{ a computable permutation}\}.$$

If $\overline{P}(A) = \underline{P}(A)$, then we call this limit the intrinsic density of A and denote it by $P(A)$.

Of special interest to the first half of this dissertation is the property of having intrinsic density 0, which has been studied extensively by Astor [4],[3] in relation with other notions of smallness such as immunity. We will refer to sets that have intrinsic density 0 as *intrinsically small* to ease notation slightly. Technically finite sets meet this definition, but from here on we shall use the term to refer to infinite sets, as those are the interesting ones. We wish to study intrinsically small sets in order to use them as our error sets in an intrinsic version of asymptotic computability, which we shall discuss in Section 2.3.

One easy observation about intrinsically small sets is that there are more computable functions f such that $\overline{\rho}(f(A)) = 0$ for all intrinsically small sets A than just the computable permutations. For example, if π is a computable permutation, then $2 \cdot \pi$ is not a computable permutation but the image of any intrinsically small set under it still has density 0. The following definition captures the idea of classes of functions preserving smallness.

Definition 1.8. For a class $\mathcal{F}$ of (partial) computable functions from ω to ω, we say that $A \subset \omega$ is *small for* $\mathcal{F}$ if $\overline{\rho}(f(A)) = 0$ for every $f \in \mathcal{F}$.

Notice that A is intrinsically small if and only if it is small for computable permutations.

For the second half of this dissertation, we shall be primarily interested in sets of intermediate intrinsic density, i.e. sets A with $P(A) \in (0, 1)$. Using the representation for sets of the form $A = \{a_0 < a_1 < a_2 < \cdots < a_n < \dots\}$, it is not hard to see the following characterization of upper and lower asymptotic density which will prove helpful for studying densities other than 0:

Lemma 1.9. Let $A \subseteq \omega$ be $\{a_0 < a_1 < a_2 < \dots\}$. Then

- $\overline{\rho}(A) = \limsup_{n\to\infty} \frac{n+1}{a_n+1}$
- $\underline{\rho}(A) = \liminf_{n\to\infty} \frac{n}{a_n}$

Proof. Note that if $A \upharpoonright (n+1)$ has a 0 in the final bit, then

$$\rho_n(A) = \frac{|A \upharpoonright n|}{n} > \frac{|A \upharpoonright n|}{n+1} = \rho_{n+1}(A)$$

Therefore, to compute the upper density it suffices to check only those numbers n for which $A \upharpoonright n$ has a 1 as its last bit. Those numbers are exactly a_n+1 by the definition of a_n, and $|A \upharpoonright (a_n+1)| = n+1$. Therefore $\{\frac{n+1}{a_n+1}\}_{n\in\omega}$ is a subsequence of $\{\rho_n(A)\}_{n\in\omega}$ which dominates the original sequence, so $\overline{\rho}(A) = \limsup_{n\to\infty} \rho_n(A) = \limsup_{n\to\infty} \frac{n+1}{a_n+1}$.

Similarly, to compute the lower density it suffices to check only the numbers n such that the final digit of $A \upharpoonright n$ is a 0, but the final digit of $A \upharpoonright (n+1)$ is a 1. (That is, if there is a consecutive block of zeroes in the characteristic function of A, we only need to check the density at the end of the block when computing lower density, as each intermediate point of the zero block has a higher density than the end.) These numbers are exactly a_n by definition, and $|A \upharpoonright a_n| = n$. Therefore $\{\frac{n}{a_n}\}_{n\in\omega}$ is a subsequence of $\{\rho_n(A)\}_{n\in\omega}$ which is dominated by the original sequence, so $\underline{\rho}(A) = \liminf_{n\to\infty} \rho_n(A) = \liminf_{n\to\infty} \frac{n}{a_n}$. □

Interestingly, intrinsic density turns out to be a robust measure of unpredictability. In fact, we shall see below that it is equivalent to injection stochasticity. If a set X has intrinsic density, then we cannot computably shrink or enlarge parts of it with a permutation to change the density. If we knew where elements of X could be found, then we could build a permutation that sent them to a set of density 1 or 0. This intuition has a formal counterpart: Astor [3] proved that any nontrivial set with intrinsic density must be of high or DNC degree, i.e. must be sufficiently noncomputable.

A natural question to ask is what reals in the unit interval can be achieved as the intrinsic density of some set. On one hand, this question has a straightforward answer: as seen in Proposition 1.19 below, appealing to randomness will yield intrinsic density r for any $r \in (0, 1)$. However, as intrinsic density is itself a poor notion of randomness (for example, we shall show that if $P(A) = r$, then $P(A \oplus A) = r$), we would like to have a better understanding of the sets with intermediate intrinsic density that is not merely given by the existence of a much stronger set. Our goal shall be to develop technology for building examples of sets with different defined intrinsic densities (and other notions of density or stochasticity as well) that isn't reliant on randomness. Unfortunately, computable coding methods cannot do this, an observation that we will formalize in Section 3.1. This motivates our development of new tools for noncomputable coding, the `into` and `within` operations on sets, which we shall introduce in Section 1.1.4. These operations shall turn out to be highly effective at coding sets in noncomputable fashion for multiple notions of density, starting with intrinsic density in Section 3.2 and MWC density in Section 3.3.

A critical proof technique for intrinsic density will involve proving that two sets A and B cannot have different intrinsic densities by creating a computable permutation which sends A to B modulo a set of density zero. The following lemma shows that if we can do this, then the density of the image of A is the same as the density of B, and therefore that they cannot have different intrinsic densities.

Lemma 1.10. If $\overline{\rho}(H) = 0$, then $\overline{\rho}(X \setminus H) = \overline{\rho}(X \cup H) = \overline{\rho}(X)$ and $\underline{\rho}(X \setminus H) = \underline{\rho}(X \cup H) = \underline{\rho}(X)$.

Proof. Notice by definition that

$$\rho_n(X) = \rho_n(X \setminus H) + \rho_n(X \cap H).$$

Therefore,

$$\overline{\rho}(X) = \limsup_{n\to\infty} \rho_n(X) = \limsup_{n\to\infty} \rho_n(X \setminus H) + \rho_n(X \cap H).$$

By subadditivity of the limit superior,

$$\overline{\rho}(X) \leq \limsup_{n\to\infty} \rho_n(X \setminus H) + \limsup_{n\to\infty} \rho_n(X \cap H).$$

As $\overline{\rho}(H) = 0$ and $X \cap H \subseteq H$,

$$\overline{\rho}(X) \leq \limsup_{n\to\infty} \rho_n(X \setminus H) = \overline{\rho}(X \setminus H).$$

However, $\overline{\rho}(X \setminus H) \leq \overline{\rho}(X)$ because $X \setminus H \subseteq X$, so $\overline{\rho}(X) = \overline{\rho}(X \setminus H)$ as desired.

The argument for the union and the argument for lower density are functionally identical. (For the union we use $X \cup H$, X, and $H \setminus X$ in place of X, $X \setminus H$, and $X \cap H$ respectively.) □

1.1.3 Stochasticity and Randomness

As hinted above, intrinsic densities between 0 and 1 are linked to stochasticity and randomness. Here we shall provide a brief review of these from the perspective of computability theory. Stochasticity and randomness are closely related notions which also measure unpredictability (how much information an observer lacks), and turn out to have strong ties to intrinsic density. Stochasticity represents the idea that we cannot select bits from an infinite sequence of 0's and 1's in such a way that the ratio of 1's to the number of bits is not the same as the ratio for the original sequence.

One can think of this as having an infinite sequence X of 0-1-valued coins, where we also think of X as a set under the identification $X = \{n : \text{the } n\text{-th coin is 1-valued}\}$. We try to use some selection process to pick coins from X to build a new sequence of coins Y with $\rho(Y) \neq \rho(X)$. If we are successful, then X is not stochastic. Changing the ways we are allowed to select coins gives us different notions of stochasticity. We review the noteworthy notions of stochasticity from the literature.

A monotone *selection function* is a partial function $f : 2^{<\omega} \to \{0, 1\}$. That is, f looks at a finite binary string and decides if it wants to select (i.e. return 1) the following bit or not based on the previous bits. A selection function may actively decline to select a bit (i.e. return 0), or simply never make a decision (i.e. diverge). Given a selection function f, it induces a partial map $\hat{f} : 2^\omega \to 2^\omega$

that is defined via $\hat{f}(A) = \{n : f(A \upharpoonright n) \downarrow = 1\}$ for all A . (We shall abuse notation and allow f to represent both a monotone selection function and the induced map $\hat{f}$ on Cantor space.) We say A is von Mises-Wald-Church (MWC) stochastic for r if either $f(A)$ does not exist (i.e., f selects only finitely many bits on A) or $\rho(\{n : p_{f(A)}(n) \in A\}) = r$ for all computable monotone selection functions. If we restrict this to only the total f, then the corresponding notion is called Church stochasticity. In both cases, we may use the results of the first n bits to computably determine whether or not we want to select the $(n+1)$-st bit, but all of the bits we select must be counted in order. (It is straightforward to check that this is the same definition as one would find in Downey-Hirschfeldt [8], however it is expressed in the notation of densities.)

Using our coin analogy, for Church stochasticity, all of the coins have been covered by cups. We must choose whether or not to add the first coin to our new sequence before looking under any cups. Then we look under the first cup and check the value, and we use this information moving forward. Having revealed the first n coins, we must computably choose whether or not to select the $(n+1)$-st coin (i.e., determine if we think it is 1-valued) prior to revealing it.

For MWC stochasticity, at each step we provide a program that will decide whether or not to select the $(n+1)$-st coin based on the results of the first n coins. We then run the program and look under the cup. We do not need to wait for the program to halt (and it may never halt) before continuing on to the next coin, but we can never go back and feed the program more information or change it in any way. Even though our selection process for the n-th coin may halt after the value of the n-th coin is known, it could not have had access to that information in its calculation.

Another historically important notion of stochasticity is Kolmogorov-Loveland stochasticity, or KL-stochasticity. This notion is similar to MWC-stochasticity, however we are allowed to select coins out of order and are not required to view all of them. Formally, a *finite assignment* is an element σ of $(\omega \times \{0,1\})^{<\omega}$ satisfying that each $n \in \omega$ appears in the first element of at most one pair of σ, that is a finite sequence of pairs (a, b) for distinct natural numbers a and not necessarily distinct $b \in \{0,1\}$. A finite assignment encodes finitely many bits of a set A much like a finite binary string; however, finite assignments are not required to define initial segments of A. We define FA to be the set of all finite assignments. A *scan rule* is a partial function $s : FA \to \omega$ which satisfies that for all $\sigma \in FA$, neither $(s(\sigma), 0)$ nor $(s(\sigma), 1)$ is in σ. A scan rule determines how we select the

next bit based on the information of finitely many bits, possibly out of order: having seen the a-th bit is a b for each $(a, b) \in \sigma$, $s(\sigma)$ is the index of the next bit we wish to check. Given a set A and a scan rule s, we recursively define $s_A(n)$ to represent the result of the first n bits of A selected by s: $s_A(0) = \emptyset$ and

$$s_A(n+1) = s_A(n)^\frown(s(s_A(n)), A(s(s_A(n)))).$$

We consider pairs (s, c) of a scan rule s and a partial choice function $c : FA \to \{0, 1\}$. The scan rule determines how to look at the bits, and the choice function c determines which bits to select, i.e. if $c(\sigma) = 1$ then we would like to count the $s(|\sigma| + 1)$-th bit. Revisiting the coin analogy, the coins are once again hidden under cups. The scan rule s determines the order in which we look under the cups, and the choice function c determines whether we want to select the next coin based on those already revealed.

A pair (s, c) induces a pair of partial maps $s_{KL}, c_{KL} : 2^\omega \to 2^\omega$ via

$$c_{KL}(A) = \{n : c(s_A(n)) = 1\}$$

and

$$s_{KL}(A) = \{n : A(s_A(n)) = 1\}.$$

That is, $c_{KL}(A)$ represents the set of bits of A selected by c and $s_{KL}(A)$ represents those bits rearranged into the order in which they were selected. Then A is Kolmogorov-Loveland stochastic, or KL-stochastic, for r if for all computable choice functions c and computable scan rules s either $s_{KL}(A)$ or $c_{KL}(A)$ does not exist (i.e., s and c combine to only select finitely many bits) or $\rho(\{n : p_{s_{KL}(A)}(n) \in c_{KL}(A)\}) = r$.

If the process of selection is uniform in regards to the input set, i.e. the order we select coins in does not change in relation to the value of those coins, then we obtain the weaker notion of injection stochasticity. Formally, a set A is injection stochastic for r if $\rho(f^{-1}(A)) = r$ for all total computable injective f. Permutation stochasticity, as expected, is the subclass where f is required to be a permutation. Using this definition, Astor first observed the following:

> **Proposition 1.11** (Astor [4] Lemma 4.2)**.** A set A is r-injection stochastic if and only if it is r-permutation stochastic.

Proof. If A is r-injection stochastic, it is trivially r-permutation stochastic.

Suppose that A is r-permutation stochastic. Then $\rho(\pi(A)) = r$ for every computable permutation π. Let f be a total computable injective function and let $F = \{n! : n \in \omega\}$. Define π_f via $\pi_f(n) = f(n)$ if $n \notin F$ and $f(n)$ is not in $\pi_f([0, n))$, and the least element of the complement of $\pi_f([0, n))$ otherwise. As π_f is a computable permutation, so is π_f^{-1} and thus $\rho(\pi_f^{-1}(A)) = r$.

Now notice that $\pi_f^{-1}(A) \upharpoonright n$ differs from $f^{-1}(A) \upharpoonright n$ by at most $2|F \upharpoonright n|$, as there can only be disagreement on F and $f^{-1}(\pi_f(F))$. In fact, there are two types of disagreement. In the first, we specifically mapped $\pi_f(n!)$ to something other than $f(n!)$, which can only happen within F. In the second, k is not a factorial but $f(k) \in \pi_f([0, k))$ because of some $n! < k$. Thus the set of disagreements has density zero because F does, so

$$\rho(f^{-1}(A)) = \rho(\pi_f^{-1}(A)) = r.$$

□

It is immediate from the definition that r-permutation stochasticity is exactly intrinsic density r. Therefore, this lemma shows that r-injection stochasticity also corresponds to intrinsic density r. Unlike stochasticity, intrinsic density is defined without fixing r ahead of time. Motivated by this, we shall use $\mathcal{C}$-density r to mean $\mathcal{C}$ stochasticity for r, where $\mathcal{C}$ stands for some fixed notion of stochsticity such as injection or MWC.

While computability theory most commonly studies stochasticity with regards to $\frac{1}{2}$, stochasticity with regards to parameters other than $\frac{1}{2}$ has been studied before. For example, see Kjos-Janssen, Taveneaux, and Thapen [16]. However, our use of the term density as opposed to stochasticity is to differentiate our intentions: stochasticity is generally studied by fixing some $r \in [0, 1]$ and a notion of stochasticity and then studying the class of sets which are stochastic for r. (In the context of randomness, this corresponds to fixing a measure from the outset.) Density, on the other hand, does not fix r and studies the class of sets which are stochastic for some r. This is a larger class that often has its own interesting properties. While the same sets appear in both settings, we are really studying the class containing these sets.

We shall study the class of sets with intrinsic (injection-) density in Section 3.2, and we shall

study the class of MWC-density sets in Section 3.3 using the tools developed in Section 3.2. One important trait of Church-density is that if A has Church-density α, then $\rho(A) = \alpha$ because the selection function $\hat{1}$ which selects every bit is a total computable monotone selection function. It follows immediately from the fact that MWC-density is defined for a larger class of selection functions and therefore contains $\hat{1}$ that the same is true of MWC-density.

Closely related, randomness is well-studied and more well-known than stochasticity, so we shall only provide a cursory overview. (For a more in-depth review of randomness as well as stochasticity, see Downey-Hirschfeldt [8].) While there are many notions of randomness, we shall only need 1-Randomness, also known as Martin-Löf Randomness, for our purposes. There are many equivalent ways of defining randomness, and we shall recall two. In computability theory most randomness is studied with respect to the Lebesgue measure, so we shall start with the more familiar form before generalizing the definitions to arbitrary measures.

Definition 1.12. A martingale is a function $m : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ such that

$$m(\sigma) = \frac{1}{2}m(\sigma 0) + \frac{1}{2}m(\sigma 1)$$

for all σ. A supermartingale is a function $s : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ with

$$s(\sigma) \geq \frac{1}{2}s(\sigma 0) + \frac{1}{2}s(\sigma 1)$$

for all σ. A (super)martingale m succeeds on a set X if $\limsup_{n\to\infty} m(X \upharpoonright n) = \infty$. X is 1-Random if no computably enumerable supermartingale succeeds on it.

Martingales capture the unpredictability of random sets: we could not win arbitrarily large amounts of money betting on the bits of X in any c.e. or computable way. An alternative yet equivalent formulation of randomness is the measure-theoretic approach, which is based upon the intuition that if a set is random then it should avoid all small sets which can be described with computable approximations.

Definition 1.13. A Martin-Löf (ML) test is a sequence $\{\mathcal{U}_i\}_{i\in\omega}$ of uniformly Σ^0_1 classes with $\mu(\mathcal{U}_i) \leq 2^{-i}$ for all i. (Here μ is the usual Lebesgue measure on Cantor space.) A set X passes $\{\mathcal{U}_i\}_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$. X is 1-Random if it passes every Martin-Löf test.

While historically the study of algorithmic randomness began with respect to the Lebesgue or "fair coin" measure, much work has focused on studying randomness with respect to other measures. It is not difficult to see how Definition 1.13 generalizes to an arbitrary computable measure. By convention, all of our measures will be probability measures, i.e. the measure of Cantor space itself will always be 1.

Definition 1.14. Let ν be a computable measure on Cantor space. A ν-Martin-Löf test is a sequence $\{\mathcal{U}_i\}_{i\in\omega}$ of uniformly Σ^0_1 classes with $\nu(\mathcal{U}_i) \leq 2^{-i}$ for all i. A set X passes $\{\mathcal{U}_i\}_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$. X is 1-Random with respect to ν if it passes every ν-Martin-Löf test.

Note that effectivity concerns are all that keep one from generalizing this to arbitrary measures. Investigating ways to address this problem has proven to be a rich area of study. Given an arbitrary measure μ, Reimann and Slaman [21] defined randomness with respect to μ as being random with respect to some representation of μ. Conversely, Levin [17], Gács [10], and Hoyrup and Rojas [19] utilized the notion of uniform tests to give an alternate definition. Day and Miller [7] proved that these approaches are in fact the same.

One can generalize the equivalence of Definition 1.12 and Definition 1.13 to obtain a definition for randomness with respect to a measure for martingales to match 1.14.

Definition 1.15. Let μ be a computable measure. Given a finite binary string σ, $[\sigma] \subseteq 2^\omega$ represents the basic open set of extensions of σ, and $\mu(\sigma) = \mu([\sigma])$. A μ-martingale is a function $m : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ such that

$$\mu(\sigma)m(\sigma) = \mu(\sigma 0)m(\sigma 0) + \mu(\sigma 1)m(\sigma 1)$$

for all σ. A μ-supermartingale is a function $s : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ with

$$\mu(\sigma)s(\sigma) \geq \mu(\sigma 0)s(\sigma 0) + \mu(\sigma 1)s(\sigma 1)$$

for all σ. A μ-(super)martingale m succeeds on a set X if $\limsup_{n\to\infty} m(X \upharpoonright n) = \infty$. A set X is μ-1-Random if no computably enumerable μ-supermartingale succeeds on it.

As in the case of definition 1.14, there are some effectivity concerns in regards to non-computable measures, but they will not affect our work.

We are primarily concerned with the following special class of measures.

Definition 1.16. Let $0 < r < 1$ be a real number. The Bernoulli measure with parameter r, μ_r, is the measure on Cantor space such that for any $\sigma \in 2^{<\omega}$,

$$\mu_r(\sigma) = r^{|\{n<|\sigma| : \sigma(n)=1\}|}(1-r)^{|\{n<|\sigma| : \sigma(n)=0\}|}.$$

We say X is r-1-Random if it is μ_r-1-Random.

Note that $\mu_{\frac{1}{2}}$ is the usual Lebesgue measure. More generally, μ_r is the r-biased "coin flip" measure, or the measure induced by the Bernoulli probability with parameter r. As we are only working with Bernoulli measures, we shall use the Reimann-Slaman definition of randomness for noncomputable measures and rely only on the fact that every representation of μ_r can compute r,

i.e. [21] Propositions 2.2 and 2.3.[1]

Stochasticity and randomness are closely related. Cholak asked the interesting question if they can be viewed as the same, i.e. is there a natural notion $\mathcal{C}$ of stochasticity for which $\mathcal{C}$-density r corresponds to μ_r-Randomness? Bienvenu pointed out that this is essentially answered by a result of Vovk [27] (With related work done by Bienvenu [6] proving the same for weaker notions of randomness) in the negative for any reasonably natural notion: If $\{p_i\}_{i\in\omega}$ is a sequence of reals, then the generalized Bernoulli measure ν for this sequence is given by

$$\nu(\sigma) = \prod_{\sigma(i)=1} p_i \cdot \prod_{\sigma(i)=0} (1-p_i)$$

If this sequence converges to r with $\Sigma_{i=0}^{\infty}(p_i - r)^2 = \infty$, then the ν-random sets and the r-1-Random sets are disjoint. Thus under any natural notion of stochasticity $\mathcal{C}$, the selected bits will be given by independent random variables of probability arbitrarily close to probability r for all but finitely many. Therefore both r-1-randoms and ν-randoms would have $\mathcal{C}$-density r. This illustrates a fundamental difference between randomness and stochasticity: selecting subsequences and measuring their density is a fundamentally coarser measure of unpredictability than randomness.

A slight modification of the standard proof that randomness for (super)martingales is the same as randomness for Martin-Löf tests (as found in Downey-Hirschfeldt [8] Section 6.3.1, referencing work of Ville [26] and Schnorr [23]) shows that 1-Randomness with respect to μ is equivalent to μ-1-Randomness.

Theorem 1.17 (Essentially Ville [26])**.** Let μ be a computable measure. Let m be a μ-(super)martingale.

- If $\sigma \in 2^{<\omega}$ and S is a prefix-free set of extensions of σ, then
$$\sum_{\tau\in S} \mu(\tau)m(\tau) \leq \mu(\sigma)m(\sigma).$$
- Let $R_n = \{X : \exists k\ m(X \upharpoonright k) \geq n\}$. Then $\mu(R_n) \leq \frac{m(\emptyset)}{n}$.

Proof. • Note that it suffices to only consider finite sets S, as if S is infinite and $\Sigma_{\tau\in S}\,\mu(\tau)m(\tau) > \mu(\sigma)m(\sigma)$, there is some finite subset of S also exhibiting this property.

[1] As pointed out by Bienvenu, one could avoid representations altogether: It is possible to avoid martingales completely and solely argue via tests. By a result of Kjos-Hanssen [14], the so-called *Hippocratic* r-Random sets, defined via Hippocratic ML-tests (essentially ML-tests which can be accessed without seeing information about r and μ_r), are exactly the r-1-Random sets. However, we use martingales for their similarity to selection functions. While there is a notion of Hippocratic martingale, it is unknown if it defines the same notion of randomness.

We argue by induction on $|S|$. For $|S| = 1$, let $\tau \succeq \sigma$, $\tau = \sigma\gamma$ for some $\gamma \in 2^{<\omega}$. Note by induction and the definition of a μ-(super)martingale that $\mu(\gamma)m(\tau) \leq m(\sigma)$. Therefore,

$$\mu(\tau)m(\tau) = \mu(\sigma)\mu(\gamma)m(\tau) \leq \mu(\sigma)m(\sigma).$$

Now suppose $|S| = k + 1$ and the induction hypothesis holds for all $i \leq k$. Let $\gamma \succeq \sigma$ be maximal such that $\tau \succeq \gamma$ for all $\tau \in S$. Then let $S_0 \subseteq S$ be the set of all $\tau \in S$ with $\tau \succeq \gamma 0$ and let $S_1 = S \setminus S_0$. (Note that for all $\tau \in S_1$, $\tau \succeq \gamma 1$.) Therefore, as γ is maximal such that all $\tau \in S$ are extensions of γ, both $|S_0| \leq k$ and $|S_1| \leq k$. Therefore, the induction hypothesis implies that

$$\sum_{\tau \in S_0} \mu(\tau)m(\tau) \leq \mu(\gamma 0)m(\gamma 0)$$

and

$$\sum_{\tau \in S_1} \mu(\tau)m(\tau) \leq \mu(\gamma 1)m(\gamma 1).$$

Therefore,

$$\sum_{\tau \in S} \mu(\tau)m(\tau) = \sum_{\tau \in S_0} \mu(\tau)m(\tau) + \sum_{\tau \in S_1} \mu(\tau)m(\tau) \leq \mu(\gamma 0)m(\gamma 0) + \mu(\gamma 1)m(\gamma 1).$$

By the properties of a μ-(super)martingale, we have

$$\mu(\gamma 0)m(\gamma 0) + \mu(\gamma 1)m(\gamma 1) \leq \mu(\gamma)m(\gamma)$$

and therefore

$$\sum_{\tau \in S} \mu(\tau)m(\tau) \leq \mu(\gamma)m(\gamma).$$

The base case proved that $\mu(\gamma)m(\gamma) \leq \mu(\sigma)m(\sigma)$, so this concludes the induction.

- Let S be a prefix-free set that induces the Σ^0_1-class R_n with all $\tau \in S$ satisfying $m(\tau) \geq n$. By definition,

$$\mu(R_n) = \sum_{\tau \in S} \mu(\tau).$$

As each $\tau \in S$ satisfies $m(\tau) \geq n$,

$$\sum_{\tau \in S} \mu(\tau) \leq \sum_{\tau \in S} \frac{m(\tau)}{n}\mu(\tau).$$

Finally, we may apply the first part with $\sigma = \emptyset$ to obtain

$$\sum_{\tau \in S} \frac{m(\tau)}{n}\mu(\tau) \leq \frac{\mu(\emptyset)m(\emptyset)}{n}.$$

As $\mu(\emptyset) = \mu([\emptyset]) = \mu(2^\omega) = 1$, we conclude

$$\sum_{\tau \in S} \frac{m(\tau)}{n}\mu(\tau) \leq \frac{m(\emptyset)}{n}.$$

□

Theorem 1.18 (Essentially Schnorr [23])**.** A set X is μ-1-Random if and only if it is 1-Random with respect to μ.

Proof. Let m be a c.e. μ-(super)martingale. Without loss of generality, assume $m(\emptyset) = 1$. Let

$U_n = \{X : \exists k\ m(X \upharpoonright k) \geq 2^n\}$. This is a μ-Martin-Löf test by Theorem 1.17, and it is immediate that $X \in \bigcap_{n\in\omega} U_n$ if and only if m succeeds on X.

Let $\{U_n\}_{n\in\omega}$ be a μ-Martin-Löf test with $\{S_n\}_{n\in\omega}$ the uniform sequence of c.e. prefix-free sets of finite binary strings which induces $\{U_n\}_{n\in\omega}$. We shall define c.e. μ-martingales m_n via the following procedure: If we see σ enter S_n at some stage, then add 1 to $m_n(\tau)$ for all $\tau \succeq \sigma$. For $\gamma \prec \sigma$, add $\frac{\mu(\sigma)}{\mu(\gamma)}$ to $m_n(\gamma)$ if $\mu(\gamma)$ is nonzero, and 0 otherwise. Then it is immediate from this definition that $m_e : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ is a c.e. function. Furthermore, note that it is a μ-martingale: let $\sigma \in 2^{<\omega}$. We must show that $\mu(\sigma)m_n(\sigma) = \mu(\sigma 1)m_n(\sigma 1) + \mu(\sigma 0)m_n(\sigma 1)$.

As S_n is prefix-free, if $\sigma \succeq \tau \in S_n$, then

$$\mu(\sigma 1)m_n(\sigma 1) + \mu(\sigma 0)m_n(\sigma 0) = \mu(\sigma 1) + \mu(\sigma 0) = \mu(\sigma) = \mu(\sigma)m_n(\sigma)$$

by construction. Otherwise, if $\mu(\tau) = 0$ for some $\tau \preceq \sigma$, then $\mu(\sigma) = \mu(\sigma 0) = \mu(\sigma 1) = 0$ and we are done. Therefore, we may assume $\mu(\tau) > 0$ for all $\tau \preceq \sigma$. Then

$$m_n(\sigma) = \sum_{\tau\in S_n, \tau\succ\sigma} \frac{\mu(\tau)}{\mu(\sigma)} = \frac{1}{\mu(\sigma)} \sum_{\tau\in S_n,\tau\succ\sigma} \mu(\tau)$$

by definition. Note that for $i = 0, 1$,

$$m_n(\sigma i) = \sum_{\tau\in S_n,\tau\succeq\sigma i} \frac{\mu(\tau)}{\mu(\sigma i)}$$

as if $\sigma i \in S_n$ then $m_n(\sigma i) = 1 = \frac{\mu(\sigma i)}{\mu(\sigma i)}$. Therefore,

$$\mu(\sigma 1)m_n(\sigma 1) + \mu(\sigma 0)m_n(\sigma 0) = \mu(\sigma 1)(\sum_{\tau\in S_n,\tau\succeq\sigma 1} \frac{\mu(\tau)}{\mu(\sigma 1)}) + \mu(\sigma 0)(\sum_{\tau\in S_n,\tau\succeq\sigma 0} \frac{\mu(\tau)}{\mu(\sigma 0)}).$$

Factoring out the denominators, we get

$$\frac{\mu(\sigma 1)}{\mu(\sigma 1)}(\sum_{\tau\in S_n,\tau\succeq\sigma 1} \mu(\tau)) + \frac{\mu(\sigma 0)}{\mu(\sigma 0)}(\sum_{\tau\in S_n,\tau\succeq\sigma 0} \mu(\tau)) =$$

$$(\sum_{\tau\in S_n,\tau\succeq\sigma 1} \mu(\tau)) + (\sum_{\tau\in S_n,\tau\succeq\sigma 0} \mu(\tau)) = \sum_{\tau\in S_n,\tau\succeq\sigma} \mu(\tau).$$

Thus,

$$\frac{\mu(\sigma)}{\mu(\sigma)} \sum_{\tau \in S_n, \tau \succeq \sigma 1} \mu(\tau) = \mu(\sigma) \sum_{\tau \in S_n, \tau \succ \sigma} \frac{\mu(\tau)}{\mu(\sigma)} = \mu(\sigma) m_n(\sigma).$$

Thus, m_n is a μ-martingale, and $\{m_n\}_{n \in \omega}$ is a uniformly c.e. collection of μ-martingales. Furthermore, $m_n(\emptyset) = \Sigma_{\tau \in S_n} \mu(\tau) \leq 2^{-n}$. Therefore, by a slight modification of Proposition 6.3.2 of Downey-Hirschfeldt [8], $m = \Sigma_{n \in \omega} m_n$ is a c.e. μ-martingale. Finally, it follows that m succeeds on X if and only if $X \in \bigcap_{n \in \omega} U_n$. □

Astor [4] proved that 1-Random sets have density $\frac{1}{2}$ by referring to Propositions 3.2.13 and 3.2.16 of Nies [20], which state that 1-Randoms must have density $\frac{1}{2}$ and that they are closed under permutations. In fact, the more general result that r-1-Randoms have intrinsic density r is true, and we provide a simple proof here. The techniques are simple modifications to those found in Nies [20] and Downey-Hirschfeldt [8].

Proposition 1.19. Let $r \in (0, 1)$. If X is r-1-Random, then X has intrinsic density r.[2]

Proof. We shall first show that r-random sets must have density r. This is natural when one considers the martingale approach to randomness: If we expect the ratio of ones to be larger than r, then we shall bet more of our capital on ones. If we do so carefully, then our betting strategy will succeed on sets with sufficiently large upper density. Prior work has been done studying the relationship between (martin)gales and the density of a set, especially relating to dimension. For example, see Lutz [18]. We shall give a straightforward calculus proof that is sufficient for our purposes. If r is not computable, then we will implicitly work relative to a given representation of μ_r, which can compute r.

Formally, we define a family of martingales such that at least one will succeed on any set with upper density greater than r. Let $0 < \alpha < 1 - r$ be rational and consider the martingale $M_\alpha : 2^{<\omega} \to \mathbb{Q}$ defined via:

- $M_\alpha(\emptyset) = 1$
- $M_\alpha(\sigma 0) = (1 - \frac{\alpha}{1-r}) M_\alpha(\sigma)$
- $M_\alpha(\sigma 1) = (1 + \frac{\alpha}{r}) M_\alpha(\sigma)$

It is immediate that M_α is a computable r-martingale from definition. If we let n_σ denote $|\{k <$

[2]This also holds for computable and Schnorr randomness, however the proofs are more complex and outside our purview for this paper. For example, see Nies [20] 7.6.24 and 3.5.21.

$|\sigma| : \sigma(k) = 1\}|$, then we see that

$$M_\alpha(\sigma) = (1 + \frac{\alpha}{r})^{n_\sigma}(1 - \frac{\alpha}{1-r})^{|\sigma|-n_\sigma}.$$

Let $r < \epsilon \leq 1$. If $\rho_{|\sigma|}(\sigma) \geq \epsilon$, then $n_\sigma \geq \epsilon|\sigma|$ and

$$M_\alpha(\sigma) \geq (1 + \frac{\alpha}{r})^{\epsilon|\sigma|}(1 - \frac{\alpha}{1-r})^{(1-\epsilon)|\sigma|} = ((1 + \frac{\alpha}{r})^{\epsilon}(1 - \frac{\alpha}{1-r})^{1-\epsilon})^{|\sigma|}.$$

Notice that for a fixed ϵ, an exercise in calculus shows that α can be chosen such that $(1 + \frac{\alpha}{r})^{\epsilon}(1 - \frac{\alpha}{1-r})^{1-\epsilon} > 1$: As $\alpha < 1 - r$, $1 - \frac{\alpha}{1-r} > 0$, so we can take the logarithm. $(1 + \frac{\alpha}{r})^{\epsilon}(1 - \frac{\alpha}{1-r})^{1-\epsilon} > 1$ if and only if

$$\epsilon \log(1 + \frac{\alpha}{r}) + (1 - \epsilon) \log(1 - \frac{\alpha}{1-r}) > 0.$$

Rearranging, this occurs if and only if

$$\log(1 - \frac{\alpha}{1-r}) > \epsilon(\log(1 - \frac{\alpha}{1-r}) - \log(1 + \frac{\alpha}{r})).$$

As $1 - \frac{\alpha}{1-r} < 1$ and $1 + \frac{\alpha}{r} > 1$,

$$\log(1 - \frac{\alpha}{1-r}) - \log(1 + \frac{\alpha}{r}) < 0$$

and the previous expression can be rearranged to obtain

$$\frac{\log(1 - \frac{\alpha}{1-r})}{\log(1 - \frac{\alpha}{1-r}) - \log(1 + \frac{\alpha}{r})} < \epsilon.$$

By L'Hôpital's Rule, the limit of the left hand side as α approaches 0 is r. As $\epsilon > r$, there is α close enough to 0 such that this is true, and thus such that $(1 + \frac{\alpha}{r})^{\epsilon}(1 - \frac{\alpha}{1-r})^{1-\epsilon} > 1$ is true.

For such an α, M_α succeeds on any set X whose upper density is greater than ϵ, as this implies that there are infinitely many n such that $M_\alpha(X \upharpoonright n) \geq ((1 + \alpha)^{\epsilon}(1 - \alpha)^{1-\epsilon})^n$. Therefore, for any X with $\overline{\rho}(X) > r$, there is an $\epsilon > r$ with $\overline{\rho}(X) \geq \epsilon$. The corresponding M_α thus succeeds on X. Additionally, for any set X with lower density less than r, the same analysis can be applied to the complement. By switching the roles of $(1 + \frac{\alpha}{r})$ and $(1 - \frac{\alpha}{1-r})$ in the construction of M_α, we obtain an r-martingale which succeeds on X. Therefore any r-1-Random set must have density r.

Now we shall show that r-1-Random sets are also closed under permutation, completing the proof. Here the classical notion of martingales does not work as well, as permutations do not select bits monotonically in general as martingales do. However, it is not difficult to see that permutations preserve μ_r, so we shall prove this result using the measure notion of randomness. Theorem 1.18 ensures that this is sufficient.

Given $\sigma \in 2^{<\omega}$, consider $[\sigma] = \{X \in 2^\omega : \sigma \preceq X\}$. For π a computable permutation, let

$$[\pi(\sigma)] = \{X \in 2^\omega : X(\pi(n)) = \sigma(n) \text{ for all } n < |\sigma|\}.$$

Notice that $[\pi(\sigma)]$ is open. Furthermore, let $k = max_{n<|\sigma|}\{\pi(n)\}$. Then

$$P_\sigma = \{\tau \in 2^{k+1} : \tau(\pi(n)) = \sigma(n) \text{ for all } n < |\sigma|\}$$

is a prefix-free set which defines $[\pi(\sigma)]$. Then for all σ, it follows from the definition of $[\pi(\sigma)]$ that

$$\mu_r([\pi(\sigma)]) = \sum_{\tau \in P_\sigma} \mu_r(\tau) = \mu_r(\sigma) \sum_{\gamma \in 2^{k+1-|\sigma|}} \mu_r(\gamma) = \mu_r([\sigma]).$$

If $\{\mathcal{U}_i\}_{i\in\omega}$ is a μ_r-Martin-Löf test, then let $\mathcal{V}_i$ be defined via

$$\mathcal{V}_i = \bigcup_{\sigma \in U_i} [\pi(\sigma)].$$

By the above, $\mu_r(\mathcal{V}_i) = \mu_r(\mathcal{U}_i)$, so $\{\mathcal{V}_i\}_{i\in\omega}$ is also a μ_r-Martin-Löf test because π is computable. A set X passes $\{\mathcal{U}_i\}_{i\in\omega}$ if and only if $\pi(X)$ passes $\{\mathcal{V}_i\}_{i\in\omega}$ by definition. Therefore if Y is not r-1-Random, then $\pi^{-1}(Y)$ is not r-1-Random either. Thus, the r-1-Randoms are closed under computable permutation as desired. □

1.1.4 Noncomputable Coding

The standard techniques of combining sets in computability theory will not prove sufficient for our purposes, as we shall briefly discuss in Section 3.1. This failure arises from the fact that operations like the join and the Cartesian product are computable. To that end, we shall turn to noncomputable coding methods to combine sets. By this we mean methods of combining two sets A and B such that the input sets cannot necessarily be recovered computably from the output set, but can be recovered using some oracle weaker than $A \oplus B$.

The *symmetric difference*, defined via $A \triangle B = \{n : A(n) \neq B(n)\}$, is one such noncomputable coding method. Given $A \triangle B$, we cannot computably determine A or B in general. However, A and $A \triangle B$ together can compute B, and B with $A \triangle B$ can compute A.

The following coding methods are natural and computable in A and B, but do not allow us to recover A or B easily, and so do not fall prey to the problems that computable methods do. The idea is quite straightforward, and has been used informally by others such as Jockusch and Astor.

Definition 1.20. Let A and B be sets of natural numbers.

- $B \triangleright A$, or B into A, is
$$\{a_{b_0} < a_{b_1} < a_{b_2} < \dots\}$$
That is, $B \triangleright A$ is the subset of A obtained by taking the "B-th elements of A."
- $B \triangleleft A$, or B within A, is
$$\{n : a_n \in B\}$$
That is, $B \triangleleft A$ is the set X such that $X \triangleright A = A \cap B$.

With $B \triangleright A$, we are simply thinking of A as a copy of ω as a well-order and $B \triangleright A$ is the subset corresponding to B under the order preserving isomorphism between A and ω. Notice that A and $B \triangleright A$ together can compute B. The intuition for why this might work for our purposes is that if a computable permutation on ω could change the size of a copy of B living inside A, then it must have been able to change the size of B or A to begin with. We shall see below that this intuition is correct and $B \triangleright A$ will work elegantly with intrinsic density, multiplying the intrinsic densities of A and B so long as some conditions are met.

We first make a few elementary observations:

- For all A, $A = A \triangleright \omega = \omega \triangleright A = A \triangleleft \omega$.
- For all A and B and any i, a_i is either in B or $\overline{B}$. Therefore i is either in $B \triangleleft A$ or $\overline{B} \triangleleft A$ respectively, so $(B \triangleleft A) \sqcup (\overline{B} \triangleleft A) = \omega$.
- If A is intrinsically small, then so is $X \triangleright A$ for any X, as intrinsic smallness is closed under subsets. The same is not true for $X \triangleleft A$, as in general it is not necessarily a subset of A or X.
- If $B \cap C = \emptyset$, then $(B \triangleright A) \cap (C \triangleright A) = \emptyset$. Furthermore, $A = (X \triangleright A) \sqcup (\overline{X} \triangleright A)$.
- A set A has MWC-density r if $\rho(A \triangleleft f(A)) = r$ for all partial computable monotone selection functions f.
- $\triangleright$ is associative, i.e. $B \triangleright (A \triangleright C) = (B \triangleright A) \triangleright C$: By definition, $(A \triangleright C) = \{c_{a_0} < c_{a_1} < c_{a_2} < \dots\}$ and thus
$$B \triangleright (A \triangleright C) = \{c_{a_{b_0}} < c_{a_{b_1}} < c_{a_{b_2}} < \dots\}$$

Similarly, $(B \triangleright A) = \{a_{b_0} < a_{b_1} < a_{b_2} < \dots\}$, and therefore by definition

$$(B \triangleright A) \triangleright C = \{c_{a_{b_0}} < c_{a_{b_1}} < c_{a_{b_2}} < \dots\}$$

- $\triangleleft$ is not associative: Consider the set of evens E, the set of odds O, and the set N of evens that are not multiples of 4. Then

  $$(O \triangleleft N) \triangleleft E = \emptyset \triangleleft N = \emptyset$$

  However,

  $$O \triangleleft (N \triangleleft E) = O \triangleleft O = \omega$$

- $\triangleright$ and $\triangleleft$ do not associate with each other in general:

  $$B \triangleright (A \triangleleft (B \triangleright A)) = B \triangleright \omega = B$$

  but

  $$(B \triangleright A) \triangleleft (B \triangleright A) = \omega$$

  Similarly, $B \triangleleft (A \triangleright B) = \omega$, but $(B \triangleleft A) \triangleright B$ is a subset of B.

1.2 Roadmap

In this dissertation we shall embark on an exploration of intrinsic density and its relationships with asymptotic computability and unpredictability. The first half of this dissertation will be devoted to intrinsic smallness. We shall explore both the properties of these sets as a class and study some applications to the realm of asymptotic computability. In Section 2.1, we shall explore which classes of functions $\mathcal{F}$ have the property that every intrinsically small set is small for $\mathcal{F}$. We will then briefly consider intrinsic smallness as an immunity notion in Section 2.2 and show that it is separate from hyperimmunity everywhere in the Turing degrees. That is, for every hyperimmune set there is a Turing equivalent set that is hyperimmune and not only not intrinsically small, but as "big" as it can possibly be. In Section 2.3, we then turn our attention to the intrinsic variant of asymptotic computability. The four different notions of error sets in classical asymptotic computability combine with four competing notions of uniformity to yield sixteen separate categories of intrinsic computability. We prove some separations and equivalences between them. We shall conclude the first half of this dissertation by studying the relativization of intrinsic density in Section 2.4.

We obtain every real in the unit interval as the intrinsic density of some set through randomness. However, as mentioned, there is a large gap between intrinsic density and randomness. We would like to construct or find sets with arbitrary intrinsic density without needing to appeal to full randomness to better understand them: the properties of random sets have been well studied, whereas much less is known about stochastic sets. Our main goal in the second half is to construct

new technology for generating sets of different densities for both intrinsic density and MWC-density to develop a base for further work exploring the stochastic sets. Not only do we want to better understand each notion, but we'd also like to study separations between them. This is analagous to the study of differences between various notions of randomness, for example, computable randomness and Martin-Löf randomness. We shall briefly show the limitations of computable methods in Section 3.1. In Section 3.2, we shall apply noncomputable coding operations, `into` and `within`, to develop new sets with prescribed intrinsic density from old ones. Using the `into` operation, we will then be able to construct a set of intrinsic density r computable from r and any $\frac{1}{2}$-Martin-Löf random. Notably, for almost all r this constructed set will not be able to compute a μ_r-random, something that previously had no known example. Finally, Section 3.3 will attempt to apply the technology from Section 3.2 to the class of sets with MWC density, achieving the same result for r a finite sum of powers of two.

Our strategy is to find some process that takes a set A of intrinsic density α and a set B of intrinsic density β and codes A and B in such a way that we are left with a set that has new intrinsic density obtained as some function of α and β. However, we shall show in Section 3.1 that we cannot hope for this process to be computable in a way that allows us to recover the original sets, as intrinsic density was defined with the intention of blocking computable coding in the setting of asymptotic computability. We shall prove that the `into` and `within` operations are able to achieve this in Section 3.2 and combine them with other set operations to create more sophisticated constructions.

CHAPTER 2

INTRINSIC SMALLNESS

2.1 Functions and Intrinsic Density

Our work here was motivated by the desire to answer basic structural questions about intrinsic smallness. For example is the join of two intrinsically small sets intrinsically small? While this may seem obvious on its surface, the technical details need to be handled with care. Our techniques in this section will enable us to prove this. We shall see a much more difficult proof of the stronger fact that $P(A \oplus B) = r$ if and only if $P(A) = P(B) = r$ in Section 3.1.

We first note that not all intrinsically small sets are small for all computable functions, nor even all total computable functions. To do so, we use the following lemma:

> **Lemma 2.1.** Let X be a set of natural numbers. Suppose that $\{\mathcal{R}_e\}_{e\in\omega}$ is a collection of uniformly X-computable infinite sets. Then there is an intrinsically small set $A \leq \emptyset' \oplus X$ such that $A \cap \mathcal{R}_e \neq \emptyset$ for all e.

Proof. Note that the index set of injective partial computable functions is $\emptyset'$ computable, as the index set of noninjective partial computable functions is Σ^0_1. Therefore, there is a $\emptyset'$-computable function f such that $\varphi_{f(e)}$ is an enumeration of exactly the injective partial computable functions.

Let $A_0 = \emptyset$ and $r_0 = 0$. Given A_s, R_s, define A_{s+1}, r_{s+1} as follows: Using X as an oracle, find k the least element of R_s with $k > r_{s+1}$, which exists because R_s is infinite. Let $A_{s+1} = A_s \cup \{k\}$. We say e is *suitable* at stage s if $[0, k] \subseteq dom(\varphi_{f(e)})$ and $[0, 2\text{max}(\varphi_{f(e)}(A_{s+1})] \subseteq \text{range}(\varphi_{f(e)})$. Notice that $\emptyset'$ can compute whether or not e is suitable at stage s uniformly in e and s because it can ask finitely many questions about convergence. Let

$$r_{s+1} = \max\{\varphi^{-1}_{f(e)}(i) : e < s \text{ suitable at stage } s,\ i \leq 2\text{max}(\varphi_{f(e)}(A_{s+1})\} + 1.$$

Let $A = \bigcup_{s\in\omega} A_s$. By construction, $A \cap R_s \neq \emptyset$ because an element of R_s was added at stage $s + 1$. Let $\pi = \varphi_{f(e)}$ be a computable permutation. Then π is suitable at every stage because its

domain and range are ω. Let k be the element added at stage $s+2$ for some $s > e$. Then for every $i \leq 2\max(\pi(A_{s+1}))$,

$$k > r_{s+1} > \pi^{-1}(i).$$

Therefore, $\pi(k) > 2\max(\pi(A_{s+1}))$. Thus, after finitely many elements, each element of $\pi(A)$ is more than double the previous element. It follows immediately that $\overline{\rho}(\pi(A)) = 0$. As π was an arbitrary computable permutation, A is intrinsically small. □

We can now show that there is an intrinsically small set that is not small for total computable functions.

Theorem 2.2. There is a set of intrinsic density 0 that is not small for total computable functions. That is, there is an intrinsically small set A and a total computable function f such that $\overline{\rho}(f(A)) > 0$.

Proof. As defined by Jockusch and Schupp [12], let $R_e = \{n : 2^e | n \text{ but } 2^{e+1} \not| n\}$. Define $f : \omega \to \omega$ via $f(0) = 0$ and $f(n) = e$, where $n \in R_e$. (Note that this is well-defined, as the R_e's form a partition of $\omega \setminus \{0\}$.) Then f is a total computable function.

By Lemma 2.1, there is an intrinsically small set A such that $R_e \cap A \neq \emptyset$ for all e. Then $f(A)$ is cofinite (in fact it is either ω or $\omega \setminus \{0\}$). Therefore, $f(A)$ has intrinsic density 1. (So A catastrophically fails to have density 0 under f.) □

We see from this example that the failure of injectivity allowed us to cast a wide net in search of elements of A and then group them together to create a set of large density. Below, we shall see that we cannot even limit this to finite inverse images and preserve the property of being intrinsically small. In fact, we cannot even limit this to finite inverse images with uniformly computable size.

We shall need the notion of a hyperimmune set to do this. Recall that a disjoint strong array is a collection $\{D_{f(n)}\}_{n\in\omega}$ of finite sets coded by a total computable function f and the canonical indexing of finite sets, where the $D_{f(n)}$'s are pairwise disjoint. A set X is *hyperimmune* if for every disjoint strong array f, there exists some n with $D_{f(n)} \cap X = \emptyset$.

Theorem 2.3. There is an intrinsically small set that is not small for the collection of all total computable functions f such that $f^{-1}(\{n\})$ is finite (and uniformly computable) for all n. That is, there exist an intrinsically small set A and a total computable function f such that $\overline{\rho}(f(A)) > 0$ and a total computable function g such that $g(n) = |f^{-1}(\{n\})|$ for all n.

Proof. Astor [3] proved that the Turing degrees which contain an infinite intrinsically small set are

those which are not weakly computably traceable. Kjos-Hanssen, Merkle, and Stephan [15] characterized these degrees as those which are High or DNC.

It is well-known that there is a binary tree for which all paths are of PA degree. Recall that the PA degrees are exactly the DNC_2 degrees. Therefore, by the hyperimmune-free basis theorem, there is a DNC_2 degree that is hyperimmune-free. (For a review of this information, see Soare [24].) This degree contains a set A which is intrinsically small by the result of Astor. Since A is hyperimmune free, there exists a disjoint strong array g such that $D_{g(n)} \cap A \neq \emptyset$ for all n. Without loss of generality, we can assume that $\max(D_{g(n)}) < \min(D_{g(n+1)})$ for all n. (Given a disjoint strong array g, we can construct a new one h as follows: $D_{h(0)} = D_{g(0)}$, and $D_{h(n+1)}$ is the first cell of the old array whose smallest element is larger than the largest element of $D_{h(n)}$.)

Define $f : \omega \to \omega$ as follows: If $n \in D_{g(k)}$ for some k, let $f(n) = 2k$. As f is a disjoint strong array such that $\max(D_{g(n)}) < \min(D_{g(n+1)})$, this is computable and well-defined. If $n \notin \bigcup_{k\in\omega} D_{g(k)}$, then let $f(n)$ be the least odd number not realised as $f(m)$ for some $m < n$. Therefore, f is a total computable function with $|f^{-1}(\{n\})|$ finite and uniformally computable. (If $n = 2k + 1$ is odd, then the inverse image is a singleton. If $n = 2k$ is even, then $f^{-1}(\{2k\}) = D_{g(k)}$.) Furthermore, as $D_{g(n)} \cap A \neq \emptyset$ for all n, $f(A)$ contains all even numbers. Therefore, $\overline{\rho}(f(A)) \geq \frac{1}{2}$. □

We see that it is much more difficult for a set to be small for classes of non-injective functions. However, both examples relied heavily upon the fact that the functions were not injective. By switching our focus to (mostly) injective classes of functions, we can prove some positive results. First, we provide an easy technical lemma.

Lemma 2.4. Suppose C is an infinite c.e. set. Then there exists an infinite, computable $H \subseteq C$ with $\rho(H) = 0$.

Proof. Let $\{c_i\}_{i\in\omega}$ be an enumeration of C. Then let $\{h_i\}_{i\in\omega}$ be such that $h_0 = c_0$ and given h_n, $h_{n+1} = c_j$, where j is the least index with $c_j > h_n + 2^n$. Then H is computable because it is a c.e. set with an increasing enumeration, and it clearly has density 0. □

Theorem 2.5. Suppose that A is an intrinsically small set. Then A is small for the class of total computable injective functions with computable range.

Proof. We argue by contrapositive: Suppose f is a total computable injective function with computable range, and A is a set with $\overline{\rho}(f(A)) > 0$. Then we construct a computable permutation π

such that $\overline{\rho}(\pi(A)) > 0$.

Let $H \subseteq \text{range}(f)$ be a computable set of density 0. Now define $\pi : \omega \to \omega$ as follows: If $f(n) \notin H$, $\pi(n) = f(n)$. If $f(n) \in H$, let $\pi(n)$ be the least element of $H \cup \overline{\text{range}(f)}$ not realized in the range of π by $m < n$. Then π is a computable permutation, and

$$\rho_n(\pi(A)) = \frac{|\pi(A) \upharpoonright n|}{n} \geq \frac{|f(A) \upharpoonright n| - |H \upharpoonright n|}{n} = \rho_n(f(A)) - \rho_n(H).$$

(The inequality comes from the fact that π and f agree on $f^{-1}(\text{range}(f) \setminus H)$.) Therefore, we obtain

$$\overline{\rho}(\pi(A)) \geq \overline{\rho}(f(A)) - \overline{\rho}(H) = \overline{\rho}(f(a)) > 0.$$

Therefore, π is a computable permutation for which $\overline{\rho}(\pi(A)) > 0$, so A is not intrinsically small. □

There are simpler proofs of Theorem 2.5, which do not require us to create an error set and construct a permutation. However, this proof is illustrative of the techniques we shall use for more difficult proofs.

Corollary 2.6. If A is intrinsically small and f is a total computable injective function with computable range, then $f(A)$ is intrinsically small.

Proof. This follows from Theorem 2.5. We use the fact that $\pi(f(A)) = \pi \circ f(A)$ and $\pi \circ f$ is a total computable injective function with computable range because f is. □

Corollary 2.7. If A and B are intrinsically small, then so is $A \oplus B$.

Proof. If f is the function sending n to $2n$, and g is the function sending n to $2n + 1$, then by Corollary 2.6 $f(A)$ and $g(B)$ are both intrinsically small. It is easy to check that the union of two intrinsically small sets is intrinsically small, as the permutation of the union is the union of the images under the permutation. Therefore, $A \oplus B = f(A) \cup g(B)$ is intrinsically small. □

We can improve this result. The use of H in the proof allows us to notice that we can change a subset of density 0 in the range and not suffer any consequences for preserving intrinsic smallness.

Definition 2.8. A (partial) function $f : \omega \to \omega$ is *-injective, or *almost injective*, if $\rho(\{n : |f^{-1}(\{n\})| > 1\}) = 0$. That is, a (partial) function is almost injective if the subset of the range where injectivity fails has density 0.

Theorem 2.9. Suppose that A is an intrinsically small set. Then A is small for the class of total computable *-injective functions with computable range.

Proof. We again argue by contrapositive: Suppose f is total computable *-injective function with computable range, and A is a set with $\overline{\rho}(f(A)) > 0$. Then we construct a total computable injective function g with computable range such that $\overline{\rho}(g(A)) > 0$ and invoke Theorem 2.5.

Let $H \subseteq \text{range}(f)$ be infinite, computable, and have density 0. Then define $g(n) = f(n)$ if $f(n)$ has not been realized in range(g) by some $m < n$, and to be the least element of H not realized in range(g) otherwise. Then g is injective, as by construction $g(n)$ cannot be in range$(g \upharpoonright n)$ for any n. Furthermore,

$$\rho_n(g(A)) = \frac{|g(A) \upharpoonright n|}{n} \geq \frac{|f(A) \upharpoonright n| - |H \upharpoonright n| - |\{k : |f^{-1}(\{k\})| > 1\} \upharpoonright n|}{n} =$$

$$\rho_n(f(A)) - \rho_n(H) - \rho_n(\{k : |f^{-1}(\{k\})| > 1)).$$

This gives

$$\overline{\rho}(g(A)) \geq \overline{\rho}(f(A)) - \overline{\rho}(H) - \overline{\rho}(\{k : |f^{-1}(k)| > 1\} = \overline{\rho}(f(A)) > 0. \qquad \square$$

Note that while an intrinsically small set is small for the class of total computable *-injective functions with computable range, the image under such functions need not be intrinsically small. To see this, take the set A and function f from the proof of Theorem 2.3 and let $g(n) = 2^{f(n)}$. Then g is *-injective because its entire image has density zero. However, there is a computable permutation π that maps $image(g)$ to the non-factorials and the complement to the factorials. Then $\pi \circ g(A)$ includes all but finitely many of the non-factorials and is therefore density 1.

To this point, we've seen that injectivity almost everywhere has been essential in allowing all intrinsically small sets to be small for our class of functions. However, up to this point, we've also relied heavily on knowing that the range is computable: if the range is not computable, we may potentially fill in part of the range that A would have been sent to later. In this case, we'd need to shift where the elements of A are sent, potentially making the density 0 in the process. As we'll see below, there are cases in which we can avoid this issue.

Theorem 2.10. Suppose A is a set and f is a *-injective function with $\overline{\rho}(f(A)) = q > 0$ and $\overline{\rho}(\text{range}(f)) - \underline{\rho}(\text{range}(f)) < q$. Then there is a *-injective function g with computable range such that $\overline{\rho}(g(A)) > 0$.

Proof. As range(f) is c.e., there is a computable subset H of range(f) with $\underline{\rho}(H) > \overline{\rho}(\text{range}(f)) - q$

by Downey, Jockusch, and Schupp [9]. In particular,

$$\overline{\rho}(\text{range}(f) \setminus H) \leq \overline{\rho}(\text{range}(f)) - \underline{\rho}(H) < q.$$

Define $g : \omega \to \omega$ via $g(n) = f(n)$ if $f(n) \in H$, and $g(n) = 0$ otherwise. Notice that g is *-injective, as

$$\{n : |g^{-1}(\{n\})| > 1\} \subseteq \{n : |f^{-1}(\{n\}| > 1\} \cup \{0\}.$$

Furthermore, $\text{range}(g) = H \cup \{0\}$ is computable. Lastly, notice that

$$\rho_n(g(A)) = \frac{|g(A) \upharpoonright n|}{n} \geq \frac{|f(A) \upharpoonright n| - |\{k < n : k \notin H \text{ and } k \in f(A)|}{n} \geq$$

$$\frac{|f(A) \upharpoonright n| - |(\text{range}(f) \setminus H) \upharpoonright n|}{n} = \rho_n(f(A)) - \rho_n(\text{range}(f) \setminus H).$$

We saw above that $\overline{\rho}(\text{range}(f) \setminus H) \leq \overline{\rho}(\text{range}(f)) - \underline{\rho}(H) < q$. It follows that

$$\overline{\rho}(g(A)) > \overline{\rho}(f(A)) - q = q - q = 0;$$

that is, $\overline{\rho}(g(A)) > 0$. □

Corollary 2.11. Suppose that A is an intrinsically small set. Then A is small for the class of total computable *-injective functions whose range has defined density.

Proof. We again argue by contrapositive: Suppose f is total computable *-injective function whose range has defined density, and A is a set with $\overline{\rho}(f(A)) > 0$. Then by Theorem 2.10, as $\overline{\rho}(\text{range}(f)) - \underline{\rho}(\text{range}(f)) = 0$, there is a *-injective function g with computable range such that $\overline{\rho}(g(A)) > 0$. Then the conclusion follows by Theorem 2.9. □

By the remark following the proof of Theorem 2.9, we see that the image of an intrinsically small set under a total computable *-injective function whose range has defined density need not be intrinsically small. However if we restrict ourselves to injective functions, can we recover the analogue of Corollary 2.6? We shall discuss this formally in Questions 4.1 and 4.2 at the end of this dissertation.

Corollary 2.11 can already be used in conjunction with known results. For example, Jockusch (correspondence with Astor) showed that r-maximal sets have intrinsic density (and therefore density) 1, so the image of any intrinsically small set under a computable injective function whose range is maximal is small.

2.2 Hyperimmunity and Intrinsic Smallness

We now turn our attention to hyperimmune sets, a competing notion of smallness. Astor [4] studied the connection between varying notions of immunity and intrinsic density thoroughly. In particular, it is known that hyperimmune sets have intrinsic lower density 0, and therefore that hypersimple sets have intrinsic upper density 1. (Hypersimple sets are c.e. sets whose complement is hyperimmune. Recall that hyperimmune sets are infinite by definition, so hypersimple sets are co-infinite.) One question left open in [4] (later answered by Astor in [3] using a degree argument) was whether or not a hypersimple set could have lower density 0, or at least non-1 lower density. The answer is yes, showing that hypersimple sets need not have defined density. We give a constructive proof, showing that every hypersimple set yields a Turing equivalent hypersimple set that has lower density 0. (That is, every hypersimple set has an equivalent hypersimple set which is "as small as possible.")

First, it is important to note that when studying whether or not certain properties relate to intrinsic smallness, we shall study the sets themselves rather than their degrees: coding tricks can show that every Turing degree contains a set with undefined density. In the c.e. degrees, this set can be taken to be c.e.

Proposition 2.12. Every Turing degree contains a set W with $\underline{\rho}(W) = 0$ and $\overline{\rho}(W) = 1$.

Proof. Given C, let $D = \{n! : n \in C\}$ and $W = D \cup \bigcup_{n\in\omega}((2n)!, (2n+1)!)$. Then $W \equiv_T D \equiv_T C$, and $\underline{\rho}(W) = 0$ because

$$\rho_{(2n+2)!}(W) = \frac{|W \upharpoonright (2n+2)!|}{(2n+2)!} \leq \frac{(2n+1)!}{(2n+2)!} = \frac{1}{2n+2}.$$

Conversely, $\overline{\rho}(W) = 1$ as

$$\rho_{(2n+1)!}(W) = \frac{|W \upharpoonright (2n+1)!|}{(2n+1)!} \geq \frac{(2n+1)! - (2n)!}{(2n+1)!} = 1 - \frac{1}{2n+1}.$$

Clearly if C is c.e., then so is W. □

We shall see below that additional properties on the starting set C can be recovered in W by modifying the construction.

Theorem 2.13. Let C be a hypersimple set. Then there is a hypersimple set $W \equiv_T C$ with $\underline{\rho}(W) = 0$.

Proof. As C is hypersimple, it has intrinsic upper density (and therefore upper density) 1. We cannot use the strategy from Proposition 2.12 directly, as the resulting set will not even be immune, let alone hyperimmune. To avoid this problem, we shall leave intervals of C intact and introduce gaps between the intervals in noncomputable fashion. Informally, we first wish to shift portions of C over to make large gaps, ensuring that the resulting set has lower density 0. We then leave an even larger interval of C intact (albeit shifted over finitely much) to ensure that the upper density is 1. (See Figure 2.1.)

Formally, we shall define c.e. sets H_i and gaps $[u_i, u_i + m_i)$ inductively. Let $H_0 = C$. Enumerate H_0 until there is a stage s and a number n such that we see $\rho_n(H_0) > \frac{1}{2}$, which exists because $C = H_0$ has upper density 1. Then let $u_0 = n$ and let m_0 be the least natural number such that $\frac{u_0}{u_0+m_0} < \frac{1}{2}$.

Given H_e and $[u_e, u_e + m_e)$, define H_{e+1} and $[u_{e+1}, u_{e+1} + m_{e+1})$ as follows: Define $H_{e+1} = (H_e \upharpoonright u_e) \cup (H_e^{\geq u_e} + m_e)$. (For convenience, here $X^{\geq k}$ denotes $\{n \in X : n \geq k\}$, and $X + m = \{n + m : n \in X\}$.) Enumerate H_{e+1} until we come to a stage s and a number $n > u_e + m_e$ such that $\rho_n(H_{e+1,s}) > 1 - \frac{1}{e+2}$. Then set $u_{e+1} = n$ and m_{e+1} to be the least natural number such that $\frac{u_{e+1}}{u_{e+1}+m_{e+1}} < \frac{1}{e+2}$. Finally, let H be the set with characteristic function $H(m) = \lim_{n\to\infty} H_n(m)$. Note, first off, that $\bigcup_{e\in\omega}[u_e, u_e + m_e)$ is a c.e. set with increasing enumeration, and hence, computable. Furthermore, note that H itself is c.e., as $\lim_{n\to\infty} H_n(m) = H_s(m)$ for any s with $u_s > m$. Then $\underline{\rho}(H) = 0$ as desired, as $\rho_{u_i+m_i}(H) < \frac{1}{i+2}$ for all i.

H cannot serve as the desired W: The complement contains the computable subset $\bigcup_{e\in\omega}[u_e, u_e + m_e)$, so it is not even immune, let alone hyperimmune. Instead, let $W = H \cup \bigcup_{n\in C}[u_n, u_n + m_n)$: that is, enumerate the n-th gap into W whenever n enters C. Then W is c.e., and we claim that it is hypersimple.

Recall that a set is hyperimmune if and only if its principal function is not computably bounded. Suppose that $\overline{W}$ is not hyperimmune. Then its principle function is bounded by some total computable function f. However, the total computable function g defined via $g(n) = f(n + \Sigma_{i\leq n} m_i)$ must bound $\overline{C}$: The elements of $\overline{W}$ are the elements of $\overline{C}$ shifted up plus the elements of $[u_n, u_n + m_n)$ for $n \notin C$. Therefore to bound c_n, we only need to bound the first $n + \Sigma_{i\leq n} m_i$ elements of $\overline{W}$, as this

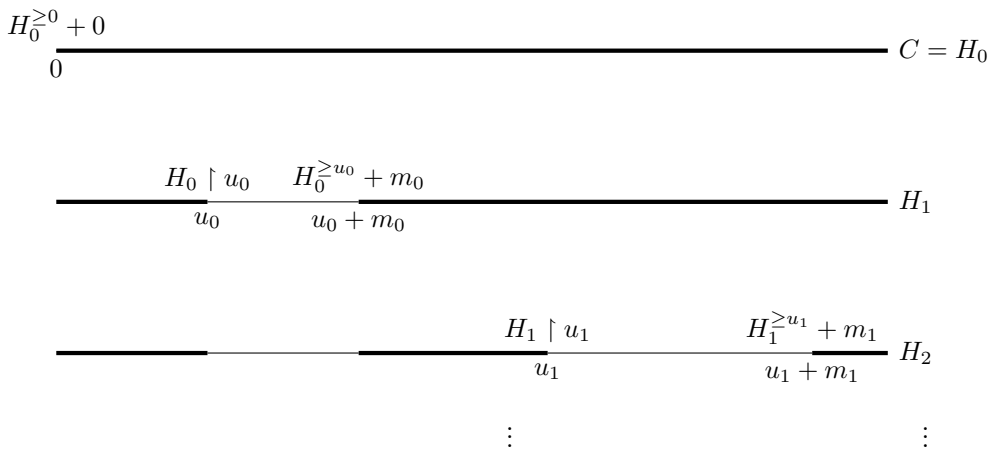

Figure 2.1. Visualization of the construction of H in Theorem 2.13

will contain at least n elements of $\overline{C}$ and no more than $\Sigma_{i \leq n} m_i$ elements introduced via gaps in W. (It is possible that this encompasses more than n elements of C, but it at least captures the n-th one.)

Thus, we have shown that W is a hypersimple set. Sine $\bigcup_{e \in \omega}[u_e, u_e + m_e]$ is computable, W can compute C by ignoring the intervals. C can clearly compute H and hence W, so they are Turing equivalent. □

By using C as an oracle rather than an enumeration of C, it is clear that this result also applies to co-hyperimmune sets in general, not just hypersimple sets.

Thus, Theorem 2.13 separates these two notions of smallness computationally. Nowhere in the Turing degrees are they equivalent. However, we can still gain some inspiration from the hyperimmune sets. Perhaps the most useful characterization of the hyperimmune sets is that a set is hyperimmune if and only if its principle function is not computably bounded. While Theorem 2.13 shows that hyperimmunity and intrinsic smallness are unrelated notions of smallness, we would like to know whether it is possible to provide a simple characterization of intrinsic smallness using principal functions, similar to the characterization of hyperimmunity. Perhaps the most natural candidate is that of weak computable traceability from [3], which does provide us with a useful test for intrinsic smallness:

> **Proposition 2.14.** Suppose that A is not intrinsically small. Then the principle function $p_A(n)$ of A is weakly computably traced, i.e. there are computable functions g and h with

$|D_{g(n)}| \leq h(n)$ for all n and $p_A(n) \in D_{g(n)}$ for infinitely many n.

Proof. Since A is not intrinsically small, there is a computable permutation π such that $\overline{\rho}(\pi(A)) = q > 0$. Define functions $h = \lambda n(n!)$ and g such that $D_{g(n)} = \pi^{-1}([0, n!))$. Then we claim that g and h witness that p_A is weakly computably traced.

To get a contradiction, suppose this is not the case. Then $p_A(k) \in D_{g(k)} = \pi^{-1}([0, k!))$ for only finitely many k. In particular, $\pi(n) \geq n!$ for all but finitely many $n \in A$. However, this implies that $\overline{\rho}(\pi(A)) = 0$, since $\rho_n(\pi(A)) \leq \frac{s+m+1}{m!}$ where s is the number of k for which $p_A(k) \in \pi^{-1}([0, k!))$ and m is the largest number with $m! \leq n$. As $\frac{s+m+1}{m!}$ approaches 0 in the limit, this contradicts the fact that $\overline{\rho}(\pi(A)) = q > 0$, so g and h must witness that p_A is weakly computably traced. □

The contrapositive of Proposition 2.14 tells us that if the principle function of A is not weakly computably traced, then A is intrinsically small. Unfortunately, Theorem 2.3 tells us that we cannot hope to reverse this in general. However, notice that the proof in fact proves a stronger statement: If A is not intrinsically small, then it is weakly computably traced with witness $h = \lambda n(n!)$. That is, if p_A is not weakly computably traced by h, then A is intrinsically small. If this can be reversed, that would characterize the intrinsically small sets. This is Question 4.5 below.

2.3 Intrinsic Computability

Having studied intrinsically small sets, we now turn our attention to their use as error sets in "almost computable" settings. Astor [4] first described four possible variations of "intrinsic" generic computability, that is "intrinsic" generic descriptions of A which ensure the existence of generic descriptions of $\pi(A)$ for all π a computable permutation. The four notions differ by how uniformly we can obtain a generic description for a given permutation. We provide the generalizations of each of these notions to the remaining three notions of asymptotic computability mentioned in Section 1.1.1, which gives us a total of sixteen separate notions. Throughout this section, x will denote an arbitrary element of {effective dense, generic, coarse, dense}. We shall begin by describing the strongest of the four notions, which is the most overtly related to our study of intrinsically small sets.

Definition 2.15. $A \subseteq \omega$ is intrinsically x-ly computable if there is an x description of A with an intrinsically small error set.

Astor originally defined this notion as strongly intrinsically x-ly computable, however we shorten

the definition for the sake of readability.

This is the most natural intrinsic variant of asymptotic computability, as it is obtained by simply requiring the error set to meet a stronger smallness condition. As we shall see, the other three notions introduced in [3] are not obtained by simply modifying the error set, but rather by introducing new restrictions on the computation.

We should verify that the intrinsically x-ly computable sets are not just the computable sets: clearly the computable sets meet this definition for any x, but are there noncomputable examples? It turns out that for the strongest notion, intrinsically effectively densely computable sets, this is not the case:

Proposition 2.16. Suppose that A is intrinsically effectively densely computable. Then A is computable.

Proof. By definition, if A is intrinsically effectively densely computable, then the error set is an intrinsically small computable set. However, no infinite computable set can be intrinsically small, as there is a computable permutation that maps it to the nonfactorials and its complement to the factorials. Therefore, the error set must be finite. As A differs from a computable set by only finitely much, it must be computable. □

Fortunately, the other three do admit noncomputable examples. For generic computability, as mentioned in [3], any c.e. set with intrinsic density 1, such as a maximal set, is intrinsically generically computable. Similarly, any set of intrinsic density 1 or 0 is intrinsically coarsely computable. Notice that any intrinsically generically computable set with defined intrinsic density must have intrinsic density 0 or 1 and thus be intrinsically coarsely computable: let φ_e be an intrinsic generic description of A. If $\{n : \varphi_e(n) \downarrow = 1\}$ is finite, then A has intrinsic density 0 because $A = \{n : \varphi_e(n) \downarrow = 1\} \cup (A \cap \overline{W_e})$ is a union of a finite set with an intrinsically small set. If this set is not finite, then it is an infinite c.e. subset of A. Therefore the absolute upper density of A is 1 because every infinite c.e. set has a computable subset, which can be mapped to the nonfactorials by a computable permutation. As A has defined intrinsic density and its absolute upper density is 1, it must have intrinsic density 1. In both cases, A is intrinsically coarsely computable. The following lemma shows that the intrsincially generically computabile sets and the intrinsically coarsely computable sets are not the same, however.

Theorem 2.17. There is an intrinsically coarsely computable set that is not intrinsically generically computable.

Proof. By Lemma 2.1, there is an intrinsically small set A such that for each infinite c.e. set W_e, there exists $a_e \in A \cap W_e$ with $a_e < a_s$ for $e < s$. That is, there is a unique designated element a_e of A for each infinite c.e. set W_e. We cannot in general use $\emptyset'$ to determine if a c.e. set is infinite, but we can use the jump to ask if there is a large enough element of W_e to continue the construction and put that into A if it exists. This may designate some elements for finite c.e. sets, but this is acceptable.

Now define $B \subseteq A$ by agreeing with A away from the a_e's and diagonalizing against the e-th turing machine using $B(a_e)$, i.e. $B(a_e) = 1 - \varphi_e(a_e)$. (Note that $\varphi_e(a_e) \downarrow$ because $a_e \in W_e$.) Then $B \subseteq A$ has intrinsic density 0 and cannot be intrinsically generically computable because it disagrees with every Turing machine with infinite domain at least once. □

The reverse separation remains open: it is easy to ensure that a given Turing function is not an intrinsic generic description by simply finding one place where it is wrong. However, to ensure that a given Turing function is not an intrinsic coarse description, we must force it to disagree on an infinite set that is not intrinsically small, which is more difficult. The natural strategy is to take an intrinsic generic description W_i, say a maximal set, and attempt to change it to diagonalize against the total functions in such a way that the description is still c.e. and the complement is still intrinsically small. The issue arises from our not being able to enumerate all of the total functions using computable indices: there is an enumeration of c.e. indices which contains exactly the computable sets (given an index e, enumerate W_e so long as the enumeration is increasing, but do not enumerate smaller elements), but there is no way to distinguish the infinite sets from the finite ones. If we know a given c.e. index e yields an infinite computable set, it is easy to wait for convergence of φ_e and diagonalize against it on an infinite computable subset of W_i, forcing φ_e to fail to be an intrinsic coarse description. However if W_e is in fact finite, then we will never see convergence, and failing to converge for the indices of finite sets will make the complement of our new enumeration no longer intrinsically small. If we give up waiting for convergence after some length of time, then there is no guarantee that an infinite computable set will ever enumerate quickly enough to be diagonalized against. This is Question 4.4 below.

Fortunately, the answer to this question resolves the remaining implications involving intrinsi-

cally densely computable sets:

Lemma 2.18. The intrinsically densely computable sets are exactly the intrinsically coarsely computable sets if every intrinsically generically computable set is intrinsically coarsely computable, and the intrinsically densely computable sets strictly contain all of the intrinsically generically computable sets and intrinsically coarsely computable sets if this is not the case.

Proof. By Theorem 2.17 there is a set B which is intrinsically coarsely computable but not intrinsically generically computable. Suppose for the first case that A is a set which is intrinsically generically computable but not intrinsically coarsely computable. An application of Corollary 2.7 tells us that $A \oplus B$ is intrinsically densely computable. However $A \oplus B$ cannot be intrinsically coarsely computable or intrinsically generically computable because any intrinsic coarse/generic description of $A \oplus B$ would necessarily yield an intrinsic coarse/generic description of A/B.

Now, we consider the second case. Suppose that every intrinsically generically computable set is intrinsically coarsely computable, and let A be intrinsically densely computable with witness φ_e. Then the set B defined via the characteristic function

$$\chi_B(n) = \begin{cases} \varphi_e(n) & n \in W_e \\ 0 & n \in \overline{W_e} \end{cases}$$

is intrinsically generically computable with witness φ_e. Therefore, it is intrinsically coarsely computable via some total witness φ_i. Thus, φ_i witnesses that A is intrinsically coarsely computable as well because the error set is contained within the union of two intrinsically small sets (the complement of W_e and the error set of φ_i on B) and thus is intrinsically small. □

The remaining three generalizations of asymptotic computation to the intrinsic setting use a separate idea: Rather than having an intrinsically small error set that ensures the existence of descriptions, we simply assert that descriptions must exist for any computable permutation. Varying the level of uniformity for these descriptions is how we reach three separate notions (Recall that $x \in \{$effective dense, generic, coarse, dense$\}$):

Definition 2.19.

- A is weakly intrinsically x-ly computable if $\pi(A)$ is x-ly computable for every computable permutation π.
- A is uniformly x-ly computable if there is a computable function $f(e, n)$ such that $\lambda n(f(e, n))$ is a(n) x description of $\varphi_e(A)$ when φ_e is a computable permutation.

- $A \subseteq \omega$ is oracle x-ly computable if there is a Turing functional Φ_i such that Φ_i^X is a(n) x description of $\varphi_e(A)$ whenever φ_e is a computable permutation and $X = \text{graph}(\varphi_e)$.

As in the case of the intrinsically x-computable sets, Astor's original definitions were "uniformly intrinsically x-ly computable" and "oracle intrinsically x-ly computable," however we shorten these definitions for readability.

It is immediate that all of the straightforward implications from asymptotic computability apply here in each of the three cases; i.e. uniformly coarsely computable sets are uniformly densely computable and so on. Furthermore, it is easy to see that for all $x \in \{\text{effective dense, generic, coarse, dense}\}$, intrinsically x-ly computabile sets are uniformly and oracle x-ly computable, where these are both weakly x-ly computable. Furthermore, albeit slightly less trivial, is the fact that oracle x-ly computable sets are uniformly x-ly computable: Given a Turing functional Φ_i witnessing that A is oracle x-ly computable, define the partial computable function $f(e, n)$ via $f(e, n) = \Phi_i^{\text{graph}(\varphi_e)}(n)$. Then the definition of oracle x-ly computable ensures that this function f witnesses uniformly x-ly computable. This means that for a fixed x, the four notions form a chain.

As noted in [4], it is unclear at first whether these notions are distinct (i.e. whether or not the chain collapses), even when restricting ourselves just to the generic case. Below we shall see that they are not distinct here, although the argument will not generalize to the coarse and dense settings. However, a slight modification of it will provide a similar but not identical result for the effective dense setting.

Theorem 2.20. Suppose that A is oracle generically computable. Then A is intrinsically generically computable.

Proof. Let Φ_i witness that A is oracle generically computable. Then define the partial computable function f as follows: Note that the set of finite binary strings σ that are initial segments of graphs of injective functions is computable. For σ in this set, let f_σ denote the partial injective function with finite range such that $\text{graph}(f_\sigma)$ is the infinite binary string obtained by adding infinitely many 0's to σ. Compute $f(n)$ by searching for such a σ with $n \in \text{range}(f_\sigma)$ and $\Phi_i^\sigma(f_\sigma(n)) \downarrow$. If one is found, define $f(n) = \Phi_i^\sigma(f_\sigma(n))$ for the first such σ. Otherwise, $f(n) \uparrow$.

First, note that $f(n) \downarrow$ implies $f(n) = A(n)$: If $f(n) \downarrow$, then there is some σ such that $\Phi_i^\sigma(f_\sigma(n)) \downarrow$. As σ is an initial segment of the graph of an injective function, σ can be extended to X where X is the graph of some computable permutation φ_e. Then as Φ_i witnesses that A is oracle

generically computable, Φ_i^X is a generic description of $\varphi_e(A)$, so $\Phi_i^X(\varphi_e(n))\downarrow$ implies

$$\Phi_i^X(\varphi_e(n)) = \varphi_e(A)(\varphi_e(n)) = A(n).$$

In particular, the finite use principle tells us that

$$A(n) = \Phi_i^X(\varphi_e(n)) = \Phi_i^\sigma(f_\sigma(n)) = f(n).$$

Thus, f is correct about A wherever it converges.

Therefore, it remains to show that the domain of f has intrinsic density 1. Notice that if φ_e is a permutation, then $\varphi_e(dom(f))$ contains $dom(\Phi_i^{\text{graph}(\varphi_e)})$, as if $\Phi_i^{\text{graph}(\varphi_e)}(k)\downarrow$, there is an initial segment σ of $\text{graph}(\varphi_e)$ with $k \in \text{range}(f_\sigma)$ that witnesses convergence, and therefore witnesses $f(\varphi_e^{-1}(k))\downarrow$. However, $\underline{\rho}(dom(\Phi_i^{\text{graph}(\varphi_e)})) = 1$ as $\Phi_i^{\text{graph}(\varphi_e)}$ is a generic description of $\varphi_e(A)$ and therefore has density 1. Thus $dom(f)$ has density 1 under every computable permutation and thus has intrinsic density 1 as desired. □

Corollary 2.21. Suppose that A is oracle effective densely computable. Then A is intrinsically generically computable.

Proof. Construct the description f of A as in the proof of Theorem 2.20, however instead of searching for convergence, search for convergence to either 0 or 1. □

As mentioned above, this argument does not in general apply to oracle coarsely computable sets and oracle densely computable sets. The issue lies in the fact that coarse and dense computation allows for mistakes, so we cannot ensure that any convergent computation is correct.

The remaining implications remain open other than the previously observed chains. The difficulty in separating these notions lies in the fact that the constructed sets cannot be described by building one error set, but rather have a different error set for each computable permutation. More importantly, these countably many computable requirements are heavily interlocked: Consider attempting to construct a weakly intrinsically generically computable set which is not weakly intrinsically coarsely computable. As an example, we may try to define an error set for the identity permutation. However, this defines the membership of the constructed set on a given c.e. set W_e. If we wish to diagonalize for a given computable permutation π, we may find that $\pi(W_e)$ has density

1, in which case we can't respect W_e and also diagonalize on a set of positive density.

2.4 Relative Intrinsic Density

We shall close this section on intrinsic smallness with a topic that transitions smoothly to our next topic on intermediate intrinsic density. The definition of intrinsic density, and by extension the definition of intrinsic smallness, admits a natural relativization:

Definition 2.22. The X-absolute upper density of $A \subseteq \omega$ is

$$\overline{P_X}(A) = \sup\{\overline{\rho}(\pi(A)) : \pi \text{ an } X\text{-computable permutation}\}$$

and the absolute lower density of A is

$$\underline{P_X}(A) = \inf\{\underline{\rho}(\pi(A)) : \pi \text{ an } X\text{-computable permutation}\}.$$

If $\overline{P_X}(A) = \underline{P_X}(A)$, then we call this limit the X-intrinsic density of A and denote it by $P_X(A)$.

It is easy to see that no infinite, co-infinite set A is A-intrinsically small. In fact, no infinite, coinfinite set A has A-intrinsic density. (One way to observe this is to note that the permutation taking A to the set W in $deg(A)$ from Proposition 2.12 is A-computable.) Furthermore, given a set A, the set of Turing degrees for which A is not intrinsically small is closed upwards and contains the cone above A. One may ask whether, for A intrinsically small, the set of degrees for which A is not intrinsically small is exactly the cone above A. The answer is no.

Proposition 2.23. There is an intrinsically small set A and a permutation $\pi \not\geq_T A$ such that $\overline{\rho}(\pi(A)) > 0$.

Proof. Let B and C be Turing incomparable intrinsically small sets. (These exist given by result of Astor saying that the degrees containing intrinsically small sets are the degrees that are high or DNC.) Then by Corollary 2.7, $A = B \oplus C$ is intrinsically small. Now let π be the B-computable permutation mapping $\{2n : n \in B\}$ to the non-factorials and the complement to the factorials. Then $\pi(B \oplus C)$ contains the non-factorials, so it has density 1. □

As a corollary, we see that given an intrinsically small set A, the set of X for which A is X-intrinsically small is not necessarily equal to the degrees strictly below A. Since B and C in the above proof are Turing incomparable, $B \oplus C$ is strictly Turing above B, but is not intrinsically small relative to B. However, it is clear that given a set A, the collection of Turing degrees of X with A X-intrinsically small is closed downwards. Must it be a Turing ideal? The following lemma shows the answer is no.

Proposition 2.24. There exist an intrinsically small set A and sets B, C with A B-intrinsically small and C-intrinsically small but not $B \oplus C$-intrinsically small. That is, the set of X for which A is X-intrinsically small is not a Turing ideal.

Proof. By the Sacks Splitting Theorem [22], there are low sets B and C such that $B \oplus C \equiv_T \emptyset'$. Therefore, a modification of Lemma 2.1 allows us to obtain a set $A \leq \emptyset'$ which is both B-intrinsically small and C-intrinsically small. (As B and C are low, $B' \equiv_T C' \equiv_T \emptyset'$, so $\emptyset'$ can enumerate the partial B and C computable injective functions and determine suitability for them.) However, A cannot be $B \oplus C$-intrinsically small because $A \leq_T \emptyset' \equiv_T B \oplus C$. □

Note that although the set of X for which A is X-intrinsically small need not be a Turing ideal, Definition 2.22 still makes sense if one considers all $\mathcal{I}$-computable permutations in a Turing ideal $\mathcal{I}$ rather than computable in a set X.

We have mentioned the fact that the intrinsically small sets are found in exactly the high or DNC degrees multiple times. It is natural to ask whether this is true under relativization. The obvious strategy is to first attempt to relativize Astor's original proof. This is successful for arithmetical degrees, but not necessarily all degrees.

Proposition 2.25 (Essentially Astor)**.** Let X be an arithmetical set. Then the Turing degrees that contain an X-intrinsically small set A are the X-high or X-DNC degrees.

Proof. We merely need to check that the proof of Corollary 2.7 from Astor [3] relativizes. A result of Kjos-Hanssen, Merkle, and Stephan [15], when relativized, says that a set A is X-weakly computably traceable if and only if it is X-high or X-DNC. (It is straightforward to check that the proof given by Downey and Hirschfeldt [8] of this result relativizes.)

We can relativize the rest of the proof of [3] Theorem 2.4, and this allows us to relativize [3] Corollary 2.5 and [3] as well. To obtain Corollary 2.7 from [3] , Astor employs the following result of Jockusch [11]: Given some property P of some sets of natural numbers, if there is an arithmetical set with the property P and P is closed under taking subsets, then the collection of Turing degrees that contain a set exhibiting P is closed upwards. The relativized form of Lemma 2.1 above yields an X'-computable X-intrinsically small set A. Since X is arithmetical, A is arithmetical, so we may apply the result of Jockusch to obtain the relativization of [3] Corollary 2.7. □

There is an obvious gap in Proposition 2.25. Specifically, can the arithmetical requirement on X be dropped? There are certainly sets X for which there are no arithmetical X-intrinsically

small sets A: If $X = \emptyset^{(\omega)}$, then X computes every arithmetical set. Therefore, there cannot be an arithmetical X-intrinsically small set.[1] An important note here is that the relativization of [3] Corollary 2.5 and Theorem 2.6 did not rely on the fact that X was arithmetical, so we already know that X-weakly computably traced sets are not X-intrinsically small, and that any non-X-weakly computably traced set computes an X-intrinsically small set for even non-arithmetical X.

Fortunately, we can use noncomputable coding methods to close this gap. We shall use the `into` operation paired with the join to show that the X-intrinsically small degrees are indeed the X-high or X-DNC degrees even for nonarithmetical X.

One downside of the `into` operation is that the degree of $A \triangleright B$ is not necessarily equal to the degree of $A \oplus B$. This is because $A \triangleright B$ cannot necessarily compute A or B. However, given B as an oracle, $A \triangleright B$ can easily compute $A = \{n : b_n \in A \triangleright B\}$. Therefore, combining the `into` operation with the join allows us to prove results about Turing degrees.

Lemma 2.26. Suppose a property $P \subseteq 2^\omega$ is closed under subsets and closed under self join, i.e. if $X \in P$ then $X \oplus X \in P$ also. Then the P-degrees are closed upwards.

Proof. Suppose A computes B, with $B \in P$. Then $B \oplus B \in P$ as P is closed under self join. Therefore, $B \oplus (A \triangleright B) \in P$ because it is a subset of $B \oplus B$. Furthermore, $B \oplus (A \triangleright B) \equiv_T A$ as A computes B and thus computes $B \oplus (A \triangleright B)$, which in turn computes A as mentioned above. Therefore $(B \oplus (A \triangleright B)$ witnesses that the degree of A is a P-degree. □

Lemma 2.26 gives us an easy proof of the classic fact that the hyperimmune degrees are closed upwards. Hyperimmune sets are closed under subsets because if $B \subseteq A$, then the principal function of B is an upper bound on the principal function of A. They are closed under self join because if $f(n)$ computably dominates $p_{(B \oplus B)}(n)$, then $f(2n)$ computably dominates $p_B(n)$. Therefore, we have met the conditions to apply Lemma 2.26.

More importantly, Lemma 2.26 allows us to close the gap in Proposition 2.25.

Corollary 2.27. For any X, the Turing degrees of X-intrinsically small sets are exactly the X-high or X-DNC degrees.

[1] A natural question arises from the appearance of $\emptyset^{(\omega)}$. A set A is said to be *arithmetically intrinsically small* if it is X-intrinsically small for every arithmetical set X. The natural question is whether there an arithmetically intrinsically small set which is not $\emptyset^{(\omega)}$-intrinsically small. The answer is yes, as $\emptyset^{(\omega)}$ can uniformly compute all of the arithmetical permutations. Therefore a modification of Lemma 2.1 allows us to construct a $\emptyset^{(\omega)}$-computable set which is arithmetically intrinsically small.

Proof. By the comments above, we only need to show that the degrees of X-intrinsically small sets are closed upwards.

By the relativized form of Corollary 2.7, the X-intrinsically small sets are closed under self-join. They are clearly closed under subsets by the definition of intrinsic smallness. Therefore, these degrees are closed upwards by Lemma 2.26, completing the proof. □

An important note about Lemma 2.26 is that it is not a strengthening of Jockusch's result: In fact, Jockusch proved this theorem to show that the cohesive degrees are closed upwards, and cohesive sets are quite easily seen to not be closed under self join. In practice, it is likely that most natural phenomena being studied will have an arithmetical example, and, thus, Jockusch's result will apply. Therefore it is likely that Lemma 2.26 will mostly be used as it was above: to prove the relativized version of a theorem where the relativization ensures there is no arithmetical example.

It is natural to ask about the Turing degrees of intrinsic density r sets for $r \in (0, 1)$. The proof that every high or DNC set computes an intrinsically small set does not generalize to intrinsic density r for arbitrary r. Even the above proof that handled the upwards closure for the intrinsically small case does not work for intrinsic density r because sets of intrinsic density r are obviously not closed under subsets. However, using a different noncomputable coding technique, we can show that the degrees of intrinsic density r sets are closed upwards. One of our main goals in the next section will be to separate the property of having intrinsic density r from being μ_r-Random. This will serve as one such separation, as the μ_r-Random degrees are not closed upwards. The proof requires a strengthening of Corollary 2.7 that we shall prove later, but we put the next result here due to its similarity with the previous one.

Theorem 2.28. The Turing degrees of the intrinsic density $r \in (0, 1)$ sets are closed upwards.

Proof. Let A compute B and let $P(B) = r$. Then, again by the result of Astor [3], B must be of high or DNC degree. Therefore, A is as well, so there must be $\hat{A} \equiv_T A$ with $\hat{A}$ intrinsically small. Thus it follows that $P(\hat{A} \triangle B) = r$, since

$$\overline{\rho}(\pi(\hat{A} \triangle B)) \leq \overline{\rho}(\pi(\hat{A} \cup B)) \leq \overline{\rho}(\pi(\hat{A})) + \overline{\rho}(\pi(B)) = \overline{\rho}(\pi(B)) = r$$

and

$$\underline{\rho}(\pi(\hat{A} \triangle B)) \geq \underline{\rho}(\pi(B \setminus \hat{A})) \geq \underline{\rho}(\pi(B)) - \underline{\rho}(\pi(\hat{A})) = \underline{\rho}(\pi(B)) = r$$

By Theorem 3.3 below, the intrinsic density r sets are closed under join and, thus, $P(B \oplus (B \triangle \hat{A})) = r$. However, $B \oplus (B \triangle \hat{A}) \equiv_T A$ because A computes both B and $\hat{A}$, and $B \oplus (B \triangle \hat{A})$ computes $\hat{A}$ via $\hat{A} = \{n : B \triangle \hat{A}(n) \neq B(n)\}$ and therefore computes A. Thus $B \oplus (B \triangle \hat{A})$ witnesses that the degree of A contains a set of intrinsic density r. □

CHAPTER 3

INTRINSIC DENSITY AND STOCHASTICITY

3.1 Computable Coding and Intrinsic Density

3.1.1 The Join

The join is the canonical operation for combining sets in computability theory. If we want to generate new sets with intrinsic density, it is natural to ask if the join does so. It is easy to show that if A has asymptotic density α and B has asymptotic density β, then $A \oplus B$ has asymptotic density $\frac{\alpha+\beta}{2}$. However, this is not the case with intrinsic density.

Proposition 3.1. If $P(A) \neq P(B)$, then $A \oplus B$ does not have intrinsic density.

Proof. We shall proceed by showing that there is a computable permutation that sends $A \oplus B$ to A modulo a set of density 0, and similarly for B. Then the upper (and lower) density of $A \oplus B$ under these permutations will match that of A and B respectively. Therefore if these densities are different, the density of $A \oplus B$ is not invariant under computable permutation.

Let $F = \{n! : n \in \omega\}$, and let $G = \overline{F}$. For any fixed computable permutation π, there is another computable permutation $\hat{\pi}$ defined via enumerating the odds onto the factorials in order and enumerating the evens onto the nonfactorials according to the ordering induced by π. That is, $\hat{\pi}(2n+1) = f_n$ and $\hat{\pi}(2n) = g_{\pi(n)}$.

Since F has density 0, Lemma 1.10 yields

$$\overline{\rho}(\hat{\pi}(A \oplus B)) = \overline{\rho}(\hat{\pi}(A \oplus B) \setminus F).$$

Since the image of the odds under $\hat{\pi}$ is a subset of F,

$$\hat{\pi}(A \oplus B) \setminus F = \hat{\pi}(A \oplus \emptyset)$$

and

$$\overline{\rho}(\hat{\pi}(A \oplus B)) = \overline{\rho}(\hat{\pi}(A \oplus \emptyset)).$$

Notice that $\hat{\pi}(A \oplus \emptyset)$ is just $\pi(A)$ with each element n increased by $|F \upharpoonright n|$. Thus,

$$\rho_n(\pi(A)) \geq \rho_n(\hat{\pi}(A \oplus \emptyset)) \geq \frac{|\pi(A) \upharpoonright n| - |F \upharpoonright n|}{n}.$$

As F is the factorials, the final expression tends to $\rho_n(\pi(A))$ in the limit, so we see that

$$\overline{\rho}(\hat{\pi}(A \oplus \emptyset)) = \overline{\rho}(\pi(A))$$

and

$$\overline{\rho}(\hat{\pi}(A \oplus B)) = \overline{\rho}(\hat{\pi}(A \oplus \emptyset)) = \overline{\rho}(\pi(A)).$$

We also have $\underline{\rho}(\hat{\pi}(A \oplus B)) = \underline{\rho}(\pi(A))$ by a nearly identical argument.

In particular, $\overline{P}(A \oplus B) \geq \overline{P}(A)$ and $\underline{P}(A \oplus B) \leq \underline{P}(A)$ because we are taking the limit superior and inferior over all computable permutations, of which $\hat{\pi}$ is but one. (Basically, $\hat{\pi}$ sends $A \oplus B$ to $\pi(A)$ modulo a set of density zero, so the intrinsic upper (lower) density of $A \oplus B$ cannot be smaller (larger) than the intrinsic upper (lower) density of A.) Reversing the use of the evens and the odds in the definition of $\hat{\pi}$, we get that the same is true for B in place of A, so $\underline{P}(A \oplus B) \leq min(\underline{P}(A), \underline{P}(B))$ and $\overline{P}(A \oplus B) \geq max(\overline{P}(A), \overline{P}(B))$. Therefore, if $P(A) \neq P(B)$, then $\underline{P}(A \oplus B) \neq \overline{P}(A \oplus B)$. □

In fact, the converse is true. The proof is much more complicated, however, and will require noncomputable coding techniques. We shall first prove a technical lemma to aid in this and later proofs.

> **Lemma 3.2.** Let $f_0, f_1, \ldots, f_k$ be a finite collection of injective computable functions and let C be a computable set. (By convention, sets are infinite unless otherwise stated. In particular, here we take C to be infinite.) Then there is a computable set $H \subseteq C$ such that $\overline{\rho}(f_i(H)) = 0$ for all i.

Proof. Let $h_0 = c_0$. Then given h_n, define h_{n+1} to be the least element c of C with $f_i(c) \geq h_n!$ for all i. Set $H = \{h_0 < h_1 < h_2 < \ldots\}$. Then $\overline{\rho}(f_i(H)) = 0$ for all i because $|f_i(H) \upharpoonright n| \leq |\{n! : n \in \omega\} \upharpoonright n|$. □

> **Theorem 3.3.** Suppose $P(A) = P(B) = \alpha$. Then $P(A \oplus B) = \alpha$.

Proof. We shall use a technical lemma to complete the proof. Let E represent the even numbers, and let O represent the odd numbers. Lemma 3.3.1 will prove that for any computable permutation π,

$$\rho(\pi(A \oplus B) \triangleleft \pi(E)) = \rho(\pi(A \oplus B) \triangleleft \pi(O)) = \alpha.$$

To show this, we give a computable permutation that sends A to $\pi(A \oplus B) \triangleleft \pi(E)$ modulo a set of density zero. We will first show that there is a computable injective function which takes A to $\pi(A \oplus B) \triangleleft \pi(E)$, and then use noncomputable coding techniques to transform it into a suitable permutation. We can use the same method to send B to $\pi(A \oplus B) \triangleleft \pi(O)$ modulo a set of density 0.

From there, we will use Lemma 3.3.1 to show that $\rho(\pi(A \oplus B)) = \alpha$, proving the theorem.

Lemma 3.3.1. Let π be a computable permutation and let A and B be as in the statement of Theorem 3.3. Then

$$\rho(\pi(A \oplus B) \triangleleft \pi(E)) = \rho(\pi(A \oplus B) \triangleleft \pi(O)) = \alpha.$$

Proof. Let $h : \pi(E) \to \omega$ send the n-th element of $\pi(E)$ to n (the inverse of the principal function), and let $d : \omega \to E$ be defined via $d(n) = 2n$. Then notice that $d(A) = A \oplus \emptyset$. Furthermore, observe that for any $X \subseteq \pi(E)$, $h(X) = X \triangleleft \pi(E)$, by the definition of h and the `within` operation. Therefore,

$$h(\pi(d(A))) = h(\pi(A \oplus \emptyset)) = \pi(A \oplus \emptyset) \triangleleft \pi(E).$$

Since $\pi(A \oplus B) \cap \pi(E) \subseteq \pi(A \oplus \emptyset)$,

$$\pi(A \oplus \emptyset) \triangleleft \pi(E) = \pi(A \oplus B) \triangleleft \pi(E).$$

Thus, $h(\pi(d(A))) = \pi(A \oplus B) \triangleleft \pi(E)$. We shall now massage h and d into permutations that preserve the relevant densities.

By Lemma 3.2, there is a computable set $H \subseteq \pi(E)$ with $\overline{\rho}(h(H)) = 0$. Now, define the computable permutation π_h via $\pi_h(n) = h(n)$ for $n \in \pi(E) \setminus H$, and have π_h enumerate $\pi(O) \sqcup H$ onto $h(H)$ in order. Similarly, define the computable permutation π_d via $\pi_d(n) = d(n)$ for $n \in \omega \setminus d^{-1}(\pi^{-1}(H))$, and have π_d enumerate $d^{-1}(\pi^{-1}(H))$ onto $O \sqcup \pi^{-1}(H)$.

Since π_d agrees with d on $\overline{d^{-1}(\pi^{-1}(H))}$, we now see that

$$\pi_d(A \setminus \pi_d^{-1}(\pi^{-1}(H))) = (A \oplus \emptyset) \setminus \pi^{-1}(H).$$

Furthermore, applying π shows that

$$\pi(\pi_d(A \setminus \pi_d^{-1}(\pi^{-1}(H)))) = \pi((A \oplus \emptyset) \setminus \pi^{-1}(H)) = \pi(A \oplus \emptyset) \setminus H.$$

Since π_h agrees with h on $\pi(E) \setminus H$ and $h(\pi(A \oplus \emptyset)) = \pi(A \oplus B) \triangleleft \pi(E)$, we have

$$\pi_h(\pi(A \oplus \emptyset) \setminus H) = (\pi(A \oplus B) \triangleleft \pi(E)) \setminus h(H).$$

Therefore, $(\pi(A \oplus B) \triangleleft \pi(E)) \setminus h(H) \subseteq \pi_h(\pi(\pi_d(A)))$ and $\pi_h(\pi(\pi_d(A))) \subseteq (\pi(A \oplus B) \triangleleft \pi(E)) \cup h(H)$.

By choice of H, $\overline{\rho}(h(H)) = 0$, so Lemma 1.10 shows that

$$\overline{\rho}(\pi_h(\pi(\pi_d(A)))) = \overline{\rho}((\pi(A \oplus B) \triangleleft \pi(E)) \setminus h(H)) = \overline{\rho}(\pi(A \oplus B) \triangleleft \pi(E))$$

and

$$\underline{\rho}(\pi_h(\pi(\pi_d(A)))) = \underline{\rho}((\pi(A \oplus B) \triangleleft \pi(E)) \setminus h(H)) = \underline{\rho}(\pi(A \oplus B) \triangleleft \pi(E)).$$

Since $P(A) = \alpha$ and $\pi_h \circ \pi \circ \pi_d$ is a computable permutation, it follows that

$$\rho(\pi(A \oplus B) \triangleleft \pi(E)) = \alpha.$$

A nearly identical argument with O in place of E and B in place of A shows that

$$\rho(\pi(A \oplus B) \triangleleft \pi(O)) = \alpha.$$

□

We shall now show that this implies that $\rho(\pi(A \oplus B)) = \alpha$. Consider $\rho_n(\pi(A \oplus B))$. By definition,

$$\rho_n(\pi(A \oplus B)) = \frac{|\pi(A \oplus B) \upharpoonright n|}{n}.$$

Since $\omega = \pi(E) \sqcup \pi(O)$,

$$\frac{|\pi(A \oplus B) \upharpoonright n|}{n} = \frac{|\pi(A \oplus B) \cap \pi(E) \upharpoonright n| + |\pi(A \oplus B) \cap \pi(O) \upharpoonright n|}{n}.$$

The latter expression can be rewritten as

$$\frac{|\pi(E) \upharpoonright n|}{|\pi(E) \upharpoonright n|} \cdot \frac{|\pi(A \oplus B) \cap \pi(E) \upharpoonright n|}{n} + \frac{|\pi(O) \upharpoonright n|}{|\pi(O) \upharpoonright n|} \cdot \frac{|\pi(A \oplus B) \cap \pi(O) \upharpoonright n|}{n}.$$

Let m be the largest number such that the m-th element of $\pi(E)$ is less than n, and let k be the analogous number for $\pi(O)$. It follows from the definition of the within operation that

$$\frac{|\pi(A \oplus B) \cap \pi(E) \upharpoonright n|}{|\pi(E) \upharpoonright n|} = \rho_m(\pi(A \oplus B) \triangleleft \pi(E))$$

and

$$\frac{|\pi(A \oplus B) \cap \pi(O) \upharpoonright n|}{|\pi(O) \upharpoonright n|} = \rho_k(\pi(A \oplus B) \triangleleft \pi(O)).$$

Therefore, we can rewrite $\rho_n(\pi(A \oplus B))$ as

$$\rho_m(\pi(A \oplus B) \triangleleft \pi(E)) \cdot \rho_n(\pi(E)) + \rho_k(\pi(A \oplus B) \triangleleft \pi(O)) \cdot \rho_n(\pi(O)).$$

Using the fact that $\rho_n(\pi(E)) + \rho_n(\pi(O)) = 1$,

$$\rho_n(\pi(A \oplus B)) = \rho_m(\pi(A \oplus B) \triangleleft \pi(E)) \cdot \rho_n(\pi(E)) + \rho_k(\pi(A \oplus B) \triangleleft \pi(O)) \cdot (1 - \rho_n(\pi(E))).$$

Rearranging, this is equal to

$$\rho_k(\pi(A \oplus B) \triangleleft \pi(O)) + \rho_n(\pi(E)) \cdot (\rho_m(\pi(A \oplus B) \triangleleft \pi(E)) - \rho_k(\pi(A \oplus B) \triangleleft \pi(O))).$$

A n goes to infinity, m and k must both go to infinity. Thus, by Lemma 3.3.1,

$$\lim_{n \to \infty} \rho_m(\pi(A \oplus B) \triangleleft \pi(E)) - \rho_k(\pi(A \oplus B) \triangleleft \pi(O)) = 0.$$

Since $\rho_n(\pi(E))$ is bounded between 0 and 1, the second term vanishes. Therefore

$$\lim_{n \to \infty} \rho_n(\pi(A \oplus B)) = \lim_{n \to \infty} \rho_k(\pi(A \oplus B) \triangleleft \pi(O)) = \rho(\pi(A \oplus B) \triangleleft \pi(O)) = \alpha$$

as desired. □

Proposition 3.1 and Theorem 3.3 can easily be generalized.

Definition 3.4. Let H be an infinite, co-infinite set. Then the H-join of A and B, denoted by $A \oplus_H B$, is

$$(A \triangleright H) \sqcup (B \triangleright \overline{H})$$

Notice that $A \oplus B = A \oplus_E B$. Furthermore, there is a computable permutation π that sends E to H and O to $\overline{H}$ in order whenever H is computable. Therefore, $\pi(A \oplus B) = A \oplus_H B$, so the generalizations of Proposition 3.1 and Theorem 3.3 follow immediately from the definition of intrinsic density.

While Proposition 3.1 is useful for allowing us to construct new sets with defined intrinsic density, we see that it is not sufficient for changing intrinsic density.

3.1.2 The Cartesian Product

Another classical candidate would be the Cartesian product $A \times B$. However, this is even less reliable than the join. Whether or not $A \times B$ even has *asymptotic* density related to the density of A and the density of B can depend on the selected pairing function. For example, if $\langle , \rangle : \omega^2 \to \omega$ is a pairing function, consider the function $f : \omega^2 \to \omega$ defined via

$$f(i, n) = \langle i - 1, n \rangle !$$

for $i > 0$ and

$$f(0, n) = s_n,$$

where S is the set of nonfactorials. Then f has all of the properties we desire in a pairing function: it is a computable bijection with computable inverse between ω^2 and ω. Using f as a pairing function, $A \times B$ (as a set of codes for pairs $\langle a, b \rangle$, $a \in A$ and $b \in B$) would have density equal to that of B if $0 \in A$ and density 0 otherwise. Removing or adding a single element from A never changes the density, let alone the intrinsic density, but we could toggle the upper density of $A \times B$ between 0 and $\overline{\rho}(B)$ by toggling whether or not 0 is in A.

Even if we fix a pairing function $\langle , \rangle$ that does respect the density of A and B, the above f shows that this will not extend to intrinsic density. Since f and $\langle , \rangle$ are both computable and have

computable inverse, there is a permutation π such that $\pi(\langle n, m\rangle) = f(n, m)$. Then $\pi(A \times B)$ will be as in the previous paragraph, so $A \times B$ cannot have intrinsic density determined by the intrinsic densities of A and B.

These methods seem like they should generalize to any attempt at "nicely" coding A and B into computable sets in such a way that we can easily recover them. This intuition will be formalized in Theorem 3.5. Therefore, we now look to apply noncomputable methods to intrinsic density. [1]

3.2 `Into`, `Within`, and Intrinsic Density

We already know that every real in the unit interval is achieved as an intrinsic density, witnessed a set with the correct type of randomness. For intrinsic density 0 and 1, randomness will only give the trivial examples $\emptyset$ and ω. However, nontrivial examples are known to exist from the work in [4] and [3], as well as the first section of this dissertation. The reliance on randomness here is not ideal. Intrinsic density is itself not a good notion of randomness because there are sets with defined intrinsic density that can be computed by arbitrarily small subsets. Let A be 1-Random and let $X_0 = A$ and $X_{n+1} = X_n \oplus X_n$. By Proposition 1.19 and Theorem 3.3, X_k will be a set of intrinsic density $\frac{1}{2}$, but $\{n : 2^k n \in X_k\} = A$, so there is a subset with density $\frac{1}{2^{k+1}}$ that computes all of X_k. However, much of the existing work on intermediate intrinsic density (and other notions of stochasticity such as MWC) is obtained solely through appealing to randomness. For example, Astor [4] proved that if A has intrinsic density α and B is 1-Random relative to A, then $A \cap B$ has intrinsic density $\frac{\alpha}{2}$. We shall build technology for exploring intrinsic density that works at the level of intrinsic density rather than at the more powerful level of randomness. For example, our techniques will allow us to generalize this result of Astor and drop the relative randomness requirement in the process. These methods will allow us to, for any $r \in (0, 1)$, construct a set of intrinsic density r which is computable from r and any $\mu_{\frac{1}{2}}$-random. For almost every r, the set we construct will be the first known example of an intrinsic density r set which cannot compute a μ_r-random set.

The methods of Section 3.1 illustrate why coding methods that enumerate a set onto a computable one are insufficient for our purposes. So long as we computably know where one of our sets

[1] We shall see in Section 3.2 that we can prove results about the union and intersection for intrinsic density, but in general they are poor tools for working with asymptotic density. $\rho(A \cap B)$ and $\rho(A \cup B)$ cannot be computed using a formula involving only $\rho(A)$ and $\rho(B)$. Furthermore, these operations do not represent a form of coding as we cannot compute A from $A \cup B$ even with the help of B for example.

A is being coded, there is a permutation which can make the resulting set look like A modulo a set of density 0, so the best case scenario is that the resulting set can have the same intrinsic density as the original sets. We therefore turn to our noncomputable coding methods.

The following theorem formalizes the observation at the end of Section 3.1: If we have some computable method of coding sets, then this method will at best preserve intrinsic density.

Theorem 3.5. Let C be computable and $P(A) = \alpha$. Then $P(A \triangleleft C) = \alpha$.

Proof. Under the map which takes c_n to n, $A \cap C$ is mapped to $A \triangleleft C$. However, unless C is ω, this is not a permutation. Using Lemma 3.2, we are able to massage this map into a permutation which takes c_n to n modulo a set of density 0. Then under this permutation, $A \cap C$ and A both go to $A \triangleleft C$ modulo a set of density 0. Therefore, if $A \triangleleft C$ did not have intrinsic density α, then A also could not, by Lemma 1.10.

Formally, assume $P(A \triangleleft C) \neq \alpha$. Suppose π is a computable permutation with $\overline{\rho}(\pi(A \triangleleft C)) > \alpha$. Let $f : C \to \omega$ be defined via $f(c_n) = n$. Then $f(A \cap C) = A \triangleleft C$:

$$A \cap C \xrightarrow{f} A \triangleleft C \xrightarrow{\pi} \pi(A \triangleleft C)$$

By Lemma 3.2, there is $H \subseteq C$ computable with $\overline{\rho}(\pi(f(H))) = 0$. Define $\pi_f : \omega \to \omega$ via $\pi_f(n) = f(n)$ for $n \in C \setminus H$, and for $n \in \overline{C} \sqcup H$, define $\pi_f(n)$ to be the least element of $f(H)$ not equal to $\pi_f(j)$ for some $j < n$. Since f agrees with π_f on $C \setminus H$,

$$\pi_f((A \cap C) \setminus H) = f(A \cap C) \setminus f(H) = (A \triangleleft C) \setminus f(H).$$

Therefore by applying π,

$$\pi(\pi_f((A \cap C) \setminus H)) = \pi((A \triangleleft C) \setminus f(H)) = \pi(A \triangleleft C) \setminus \pi(f(H)).$$

It follows from the above equality that

$$\overline{\rho}(\pi(\pi_f((A \cap C) \setminus H))) = \overline{\rho}(\pi(A \triangleleft C) \setminus \pi(f(H))).$$

Since $\overline{\rho}(\pi(f(H))) = 0$, we apply Lemma 1.10 to see that

$$\overline{\rho}(\pi(A \triangleleft C) \setminus \pi(f(H))) = \overline{\rho}(\pi(A \triangleleft C)).$$

From the fact that $(A \cap C) \setminus H \subseteq A$, we obtain

$$\overline{\rho}(\pi(\pi_f(A))) \geq \overline{\rho}(\pi(\pi_f((A \cap C) \setminus H))) = \overline{\rho}(\pi(A \triangleleft C)).$$

However, we assumed that $\overline{\rho}(\pi(A \triangleleft C)) > \alpha$, so $\overline{\rho}(\pi(\pi_f(A))) > \alpha$. Since $\pi \circ \pi_f$ is a computable permutation, this implies $P(A) \neq \alpha$.

This proves that if π is a computable permutation with $\overline{\rho}(\pi(A \triangleleft C)) > \alpha$, then $P(A) \neq \alpha$. If there is no such permutation, then there must be a computable permutation π with $\underline{\rho}(\pi(A \triangleleft C)) < \alpha$ because we assumed that $P(A \triangleleft C) \neq \alpha$. Then because

$$(\pi(A \triangleleft C)) \sqcup (\pi(\overline{A} \triangleleft C)) = \pi((A \triangleleft C) \sqcup (\overline{A} \triangleleft C)) = \pi(\omega) = \omega,$$

we have $\rho_n(\pi(\overline{A} \triangleleft C)) = 1 - \rho_n(\pi(A \triangleleft C))$ for all n. Therefore, by the subtraction properties of the limit superior,

$$\overline{\rho}(\pi(\overline{A} \triangleleft C)) \geq 1 - \underline{\rho}(\pi(A \triangleleft C)).$$

By our assumption that $\underline{\rho}(\pi(A \triangleleft C)) < \alpha$,

$$1 - \underline{\rho}(\pi(A \triangleleft C)) > 1 - \alpha.$$

Thus, $\overline{\rho}(\pi(\overline{A} \triangleleft C)) > 1 - \alpha$. We now apply the previous case to get that $P(\overline{A}) \neq 1 - \alpha$, which automatically implies $P(A) \neq \alpha$. □

We obtain an alternate proof of Proposition 3.1 as a corollary of this result.

Corollary 3.6. (Proposition 3.1) If $P(A) \neq P(B)$, then $A \oplus B$ does not have intrinsic density.

Proof. Suppose $A \oplus B$ has intrinsic density γ. Let E be the set of even numbers and O the set of odd numbers. By Theorem 3.5,

$$P((A \oplus B) \triangleleft E) = P((A \oplus B) \triangleleft O) = \gamma.$$

However $(A \oplus B) \triangleleft E = A$ and $(A \oplus B) \triangleleft O = B$, so $P(A) = P(B) = \gamma$. □

Knowing that we cannot change density using computable coding, we turn our attention to coding within noncomputable sets via `into`. This will allow us to multiply intrinsic densities.

We now make an observation about the asymptotic density of $B \triangleright A$, which will be critical for investigating the intrinsic density of sets obtained via use of the `into` operation.

Lemma 3.7.

- $\overline{\rho}(B \triangleright A) \leq \overline{\rho}(B)\overline{\rho}(A)$.
- $\underline{\rho}(B \triangleright A) \geq \underline{\rho}(B)\underline{\rho}(A)$.

Proof. By Lemma 1.9,

$$\overline{\rho}(B \triangleright A) = \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} = \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} \cdot 1 = \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} \cdot \frac{b_n+1}{b_n+1}.$$

By the submultiplicativity of the limit superior,

$$\overline{\rho}(B \triangleright A) \leq (\limsup_{n\to\infty} \frac{b_n+1}{a_{b_n}+1})(\limsup_{n\to\infty} \frac{n+1}{b_n+1}) = (\limsup_{n\to\infty} \frac{b_n+1}{a_{b_n}+1})\overline{\rho}(B).$$

Since $\{\frac{b_n+1}{a_{b_n}+1}\}_{n\in\omega}$ is a subsequence of $\{\frac{n+1}{a_n+1}\}_{n\in\omega}$, it holds that

$$\limsup_{n\to\infty} \frac{b_n+1}{a_{b_n}+1} \leq \limsup_{n\to\infty} \frac{n+1}{a_n+1} = \overline{\rho}(A).$$

Therefore, $\overline{\rho}(B \triangleright A) \leq \overline{\rho}(B)\overline{\rho}(A)$, as desired.

The case for the limit inferior is nearly identical, reversing $\leq$ to $\geq$ and using supermultiplicativity along with the corresponding identity from Lemma 1.9. □

Corollary 3.8. If $\rho(A) = \alpha$ and $\rho(B) = \beta$, then $\rho(B \triangleright A) = \alpha\beta$.

Therefore, if $B \triangleright A$ has intrinsic density, its intrinsic density must be the product of the densities of A and B. Our next goal is to prove that $B \triangleright A$ does indeed have defined intrinsic density with sufficient assumptions on A and B.

Theorem 3.9. If $P(A) = \alpha$ and $P_A(B) = \beta$, then $P(B \triangleright A) = \alpha\beta$.

Proof. The proof is similar to the proof of Theorem 3.5. However, we shall present it fully here without referring to techniques from that proof, as it is quite technical. The idea is that for any fixed computable permutation π, there is an A-computable permutation that sends B to $\pi(B \triangleright A) \triangleleft \pi(A)$ modulo a set of density 0. Therefore, if π witnesses that $B \triangleright A$ does not have intrinsic density $\alpha\beta$, $\pi(B \triangleright A)$ does not have density $\alpha\beta$, and A has intrinsic density α, then Lemma 3.7 will show that $\pi(B \triangleright A) \triangleleft \pi(A)$ does not have density β, so B does not have A-intrinsic density β.

Formally, assume $P(A) = \alpha$. Assume that $P(B \triangleright A) \neq \alpha\beta$. We shall show that $P_A(B) \neq \beta$. First, suppose that there is some computable permutation π such that $\overline{\rho}(\pi(B \triangleright A)) > \alpha\beta$. We shall let $\pi(A) = \{p_0 < p_1 < p_2 < \dots\}$. Let $f : A \to \omega$ be defined via $f(a_n) = n$ and $g : \pi(A) \to \omega$ via $g(p_n) = n$, f maps A to its indices and g maps $\pi(A)$ to its indices. Then $f(B \triangleright A) = B$ and $g(\pi(B \triangleright A)) = \pi(B \triangleright A) \triangleleft \pi(A)$:

$$\begin{array}{ccc} B \triangleright A & \xrightarrow{\pi} & \pi(B \triangleright A) \\ \downarrow f & & \downarrow g \\ B & & \pi(B \triangleright A) \triangleleft \pi(A) \end{array}$$

It follows from Lemma 3.7 that $\overline{\rho}(\pi(B \triangleright A) \triangleleft \pi(A)) > \beta$. From the definition,

$$(\pi(B \triangleright A) \triangleleft \pi(A)) \triangleright \pi(A)) = \pi(B \triangleright A)$$

and $\overline{\rho}(B \triangleright A) > \alpha\beta$ by assumption. Since $P(A) = \alpha$, we have $\overline{\rho}(\pi(A)) = \alpha$, so $\overline{\rho}(\pi(B \triangleright A) \triangleleft \pi(A)) \leq \beta$ would contradict Lemma 3.7.

From this point forward we shall let

$$X = \pi(B \triangleright A) \triangleleft \pi(A)$$

for the sake of readability.

By Lemma 3.2 relativized to A and applied to $g \circ \pi$, there is an A-computable set $H \subseteq A$ such that:

$$\overline{\rho}(g(\pi(H))) = 0$$

We shall now define permutations which preserve the properties of f and g outside of H. Define $\pi_f : \omega \to \omega$ via $\pi_f(k) = f(k)$ for $k \in A \setminus H$, and for $k \in \overline{A} \sqcup H$, let $\pi_f(k)$ be the least element

of $f(H)$ not equal to $\pi_f(m)$ for some $m < k$. Define $\pi_g : \omega \to \omega$ similarly using $\pi(A)$, $\pi(H)$, and $g(\pi(H))$ in place of A, H, and $f(H)$, respectively. Then π_f and π_g are A-computable because H, f, and g are, and they are permutations because f and g are bijections (from A and $\pi(A)$ to ω respectively) which have been modified to be total without violating injectivity or surjectivity.

Now we shall compute $\pi_g(\pi(\pi_f^{-1}(B \setminus f(H))))$. Since $f(B \triangleright A) = B$ and f agrees with π_f on $\overline{H}$,

$$\pi_f^{-1}(B \setminus f(H)) = (B \triangleright A) \setminus H.$$

In addition, it holds that

$$\pi((B \triangleright A) \setminus H) = \pi(B \triangleright A) \setminus \pi(H).$$

Since $g(\pi(B \triangleright A)) = X$ and π_g agrees with g on $\overline{\pi(H)}$,

$$\pi_g(\pi(B \triangleright A) \setminus \pi(H)) = g(\pi(B \triangleright A)) \setminus g(\pi(H)) = X \setminus g(\pi(H)).$$

Thus $\pi_g(\pi(\pi_f^{-1}(B \setminus f(H)))) = X \setminus g(\pi(H))$. Because $\overline{\rho}(g(\pi(H)) = 0$, Lemma 1.10 implies that

$$\overline{\rho}(X \setminus g(\pi(H))) = \overline{\rho}(X).$$

By the definition of X,

$$\overline{\rho}(X) = \overline{\rho}(\pi(B \triangleright A) \triangleleft \pi(A)),$$

which is greater than β by the above. Since $B \setminus f(H) \subseteq B$, we have

$$\pi_g(\pi(\pi_f^{-1}(B \setminus f(H)))) \subseteq \pi_g(\pi(\pi_f^{-1}(B))).$$

Thus,

$$\overline{\rho}(\pi_g(\pi(\pi_f^{-1}(B)))) \geq \overline{\rho}(\pi_g(\pi(\pi_f^{-1}(B \setminus f(H)))))).$$

Therefore,

$$\overline{\rho}(\pi_g(\pi(\pi_f^{-1}(B)))) \geq \overline{\rho}(\pi(B \triangleright A) \triangleleft \pi(A)) > \beta.$$

Since $\pi_g \circ \pi \circ \pi_f^{-1}$ is an A-computable permutation, $P_A(B) \neq \beta$.

We have proved that if there is some computable permutation π such that $\overline{\rho}(\pi(B \triangleright A)) > \alpha\beta$,

then $P_A(B) \neq \beta$. If there is no such permutation, then there must be a computable permutation π such that $\underline{\rho}(\pi(B \triangleright A)) < \alpha\beta$, because we assumed that $P(B \triangleright A) \neq \alpha\beta$. Since $A = (B \triangleright A) \sqcup (\overline{B} \triangleright A)$, we get $\pi(A) = \pi(B \triangleright A) \sqcup \pi(\overline{B} \triangleright A)$. Therefore

$$\overline{\rho}(\pi(\overline{B} \triangleright A)) = \overline{\rho}(\pi(A) \setminus \pi(B \triangleright A)).$$

The fact that $\rho_n(\pi(A)) = \rho_n(\pi(B \triangleright A)) + \rho_n(\pi(A) \setminus \pi(B \triangleright A))$, when combined with the properties of the limit superior with regards to subtraction, implies

$$\overline{\rho}(\pi(A) \setminus \pi(B \triangleright A)) \geq \overline{\rho}(\pi(A)) - \underline{\rho}(\pi(B \triangleright A)).$$

We know that $\overline{\rho}(\pi(A)) = \alpha$ because $P(A) = \alpha$. We assumed that $\underline{\rho}(\pi(B \triangleright A)) < \alpha\beta$, so

$$\overline{\rho}(\pi(\alpha)) - \underline{\rho}(\pi(B \triangleright A)) > \alpha - \alpha\beta = \alpha(1 - \beta).$$

All together, we have

$$\overline{\rho}(\pi(\overline{B} \triangleright A)) > \alpha(1 - \beta).$$

Thus, we can apply the first case of the proof to show that $P_A(\overline{B}) \neq 1 - \beta$. This automatically implies $P_A(B) \neq \beta$, so we are done. □

Astor [4] proved that whenever $P(A) = \alpha$ and B is 1-Random relative to A, then $P(A \cap B) = \frac{\alpha}{2}$. By Proposition 1.19, the fact that B is 1-Random relative to A implies B has A-intrinsic density $\frac{1}{2}$. This hints at a more general theorem, which we can prove as a corollary of the previous two theorems.

Corollary 3.10. If $P(A) = \alpha$ and $P_A(B) = \beta$, then $P(A \cap B) = \alpha\beta$.

Proof. By definition,

$$A \cap B = (B \triangleleft A) \triangleright A.$$

Since $P_A(B) = \beta$, Theorem 3.5 relativized to A shows that $P_A(B \triangleleft A) = \beta$. Therefore, we can apply Theorem 3.9 to A and $B \triangleleft A$ to get that

$$P((B \triangleleft A) \triangleright A) = P(A \cap B) = \alpha\beta.$$

□

Before continuing, we consider possible improvements to this theorem. It is natural to ask whether or not either of the intrinsic density requirements in the statement of Theorem 3.9 can be removed or weakened. It is immediate that we cannot drop the requirement that A has intrinsic density. Because $P_A(\omega) = 1$ for any A, ω always satisfies the requirements on B, but $\omega \triangleright A = A$, so A must have intrinsic density. Similarly, $B \triangleright \omega = B$ for any B, so B must have intrinsic density. Therefore, the only possible weakening of Theorem 3.9 would be to require $P(B) = \beta$ as opposed to $P_A(B) = \beta$. However, this fails.

Proposition 3.11. Let $P(A) = \frac{1}{2}$. Then $P(A \oplus \overline{A}) = \frac{1}{2}$, but $A \triangleright (A \oplus \overline{A})$ does not have intrinsic density.

Proof. Because $P(A) = \frac{1}{2}$ implies $P(\overline{A}) = \frac{1}{2}$, $A \oplus \overline{A}$ has intrinsic density $\frac{1}{2}$ by Theorem 3.3.

Let E represent the set of even numbers. Notice that $A \oplus \overline{A}$ contains exactly one of $2k$ or $2k+1$ for all $k \in \omega$. Therefore, the n-th element of $A \oplus \overline{A}$ is $2n$ if $n \in A$ and $2n+1$ if $n \notin A$. Thus, by definition,

$$E \triangleleft (A \oplus \overline{A}) = A.$$

By the properties of the `within` operation,

$$A \triangleright (A \oplus \overline{A}) = (E \triangleleft (A \oplus \overline{A})) \triangleright (A \oplus \overline{A}) = E \cap (A \oplus \overline{A}) = A \oplus \emptyset.$$

By Proposition 3.1, however, $A \oplus \emptyset$ does not have intrinsic density. □

We cannot extend this result to $A \oplus_H \overline{A}$ in general. Specifically, it is not always true that $H \triangleleft (A \oplus_H \overline{A}) = A$. For example, consider the set H of naturals congruent to 2 modulo 3, and let A be a set containing 0 but not containing 1. Then $0 \notin H \triangleleft (A \oplus_H \overline{A})$, because $p_{A \oplus_H \overline{A}}(0) = 1$ and $1 \notin H$. Thus, $H \triangleleft (A \oplus_H \overline{A}) \neq A$ as witnessed by 0.[2]

With these tools in hand, we may now look towards constructing a set of arbitrary intrinsic density. To do this, we would like to have a countable collection of sets that all have intrinsic density relative to each other, so that we may apply Theorem 3.9 repeatedly.

[2]Recall that Theorem 3.5 says that if $P(A) = \alpha$ and C is computable, then $A \triangleleft C$ also has intrinsic density α. It is natural to wonder if this is symmetric: does $C \triangleleft A$ have intrinsic density? The proof of Proposition 3.11 shows that it is possible for $C \triangleleft A$ to have intrinsic density. However, this is not true in general, as $C \triangleleft \omega = C$. Future work exploring this may reveal something interesting about the structure of sets with intrinsic density: let $P(A) > 0$, C be coinfinite, computable with $P(C \triangleleft A) > 0$. Such sets would witness the failure of the weak version of Theorem 3.9, as $(C \triangleleft A) \triangleright A = C \cap A$ and no coinfinite subset of a computable set can have intrinsic density greater than zero.

Lemma 3.12. There is a countable, disjoint sequence of sets $\{A_i\}_{i\in\omega}$ such that $P(A_i) = \frac{1}{2^{i+1}}$. Furthermore, the A_i's form a partition of ω.

Proof. For a given set X, $X^{[i]}$ denotes the i-th column of X, $\{n : \langle i, n\rangle \in X\}$. Let $X \subseteq \omega$ be 1-Random. Then for all i, $X^{[i]}$ is 1-Random relative to $\bigoplus_{j\neq i} X^{[j]}$. (This is essentially Van Lambalgen's Theorem [25]. See Downey-Hirschfeldt [8] Corollary 6.9.6 for the details.) Proposition 1.19, when relativized, implies that Z-1-Randoms have Z-intrinsic density $\frac{1}{2}$. In particular, taking a single 1-Random automatically gives us infinitely many mutually 1-Random sets, and, thus, infinitely many sets with intrinsic density $\frac{1}{2}$ relative to each other. Using these sets and applying Theorem 3.9, we can construct the desired sequence, where the mutual randomness ensures us that the conditions of the theorem are met.

Let $B_0 = \omega$. Given B_n, let

$$A_n = \overline{X^{[n]}} \triangleright B_n$$

and

$$B_{n+1} = X^{[n]} \triangleright B_n.$$

Note that for all i, $B_{i+1} \subseteq B_i$ and $A_i \cap B_{i+1} = \emptyset$, since $B_{i+1} = X^{[i]} \triangleright B_i$ and $A_i = \overline{X^{[i]}} \triangleright B_i$. Then for $i < j$, $A_i \cap A_j = \emptyset$ because $A_j \subseteq B_j \subseteq B_{i+1}$. Thus, the family $\{A_i\}_{i\in\omega}$ is disjoint. We now verify that $P(A_i) = \frac{1}{2^{i+1}}$ and $P(B_i) = \frac{1}{2^i}$ by induction.

For the base case, $P(B_0) = P(\omega) = 1$, and B_0 is computable. Suppose that B_i is $\bigoplus_{j<i} X^{[j]}$-computable and that $P(B_i) = \frac{1}{2^i}$. Then $B_{i+1} = X^{[i]} \triangleright B_i$ is $B_i \oplus X^{[i]}$-computable, and therefore $\bigoplus_{j<i+1} X^{[j]}$-computable. Then by the above, both $X^{[i]}$ and $\overline{X^{[i]}}$ are 1-Random relative to B_i. Therefore, by the relativization of Proposition 1.19, we get $P_{B_i}(X^{[i]}) = P_{B_i}(\overline{X^{[i]}}) = \frac{1}{2}$. Thus, by Theorem 3.9, we have

$$P(A_i) = P(\overline{X^{[i]}} \triangleright B_i) = P(\overline{X^{[i]}})P(B_i) = \frac{1}{2} \cdot \frac{1}{2^i} = \frac{1}{2^{i+1}}.$$

A nearly identical argument for $P(B_{i+1})$ verifies $P(B_{i+1}) = \frac{1}{2^{i+1}}$, which completes the induction.

Finally, suppose for the sake of contradiction that the A_i's do not form a partition of ω. Since we have already shown that the sequence is disjoint, there must exist a least m with $m \notin A_i$ for any i. Therefore, there must be some k such that m is the least element of B_n for all $n \geq k$. This

is because every m is in B_0 and B_{i+1} is a subset of B_i missing only elements of A_i. It follows that $0 \in X^{[n]}$ for all $n > k$, as $0 \in \overline{X^{[n]}}$ would imply that $m \in A_n$ since $A_n = \overline{X^{[n]}} \triangleright B_n$ and m is the least, i.e. 0-th, element of B_n. However, this means that $\{\langle n, 0\rangle : n > k\}$ is an infinite computable subset of X, which contradicts the assumption that X is 1-random since it is a basic fact that 1-random sets cannot have infinite computable subsets. Therefore, every m must be in some A_i as desired. □

Jockusch and Schupp [12] proved that asymptotic density enjoys a restricted form of countable additivity: if there is a countable disjoint sequence $\{S_i\}_{i\in\omega}$ of sets such that $\rho(S_i)$ exists for all i, and

$$\lim_{n\to\infty} \overline{\rho}(\bigsqcup_{i>n} S_i) = 0,$$

then

$$\rho(\bigsqcup_{i\in\omega} S_i) = \sum_{i=0}^{\infty} \rho(S_i).$$

The intrinsic density analog of this results follows immediately from the fact that permutations preserve disjoint unions. That is, if there is a countable disjoing sequence $\{S_i\}_{i\in\omega}$ of sets such that $P(S_i)$ exists for all i and

$$\lim_{n\to\infty} \overline{P}(\bigsqcup_{i>n} S_i) = 0,$$

then

$$P(\bigsqcup_{i\in\omega} S_i) = \sum_{i=0}^{\infty} P(S_i).$$

Note that $\lim_{n\to\infty} \overline{P}(\bigsqcup_{i>n} A_i) = 0$ must be true for any collection of sets satisfying Lemma 3.12 because $\lim_{n\to\infty} P(\bigsqcup_{i\leq n} A_i) = 1$. This, when combined with the previous lemma, will allow us to construct a set with intrinsic density r for any $r \in (0, 1)$.

> **Theorem 3.13.** For every $r \in (0, 1)$ and any 1-random set X, $r \oplus X$ computes a set with intrinsic density r.

Proof. Let $r \in (0, 1)$. Let $B_r \subseteq \omega$ be the set whose characteristic function is identified with the binary expansion that gives r, the set of all n such that the n-th bit in the binary expansion for r is 1. Let $\{A_i\}_{i\in\omega}$ be constructed from X as in the proof of Lemma 3.12, and let $X_r = \bigsqcup_{n\in B_r} A_n$. We now describe the process by which X_r is computable from $r \oplus X$. For a given m, $m \in A_n$ for some n since the A_i's form a partition of ω. X can uniformly compute the A_i's and thus compute n. Then $m \in X_r$ if and only if $n \in B_r$, which r can compute.

Now, note that

$$\lim_{n\to\infty} \overline{P}(\bigsqcup_{i\in B_r, i>n} A_i) = 0$$

because $\bigsqcup_{i\in B_r,i>n} A_i \subseteq \bigsqcup_{i>n} A_i$ and $\lim_{n\to\infty} \overline{P}(\bigsqcup_{i>n} A_i) = 0$. By the fact that countable unions sum intrinsic densities and the definition of X_r,

$$P(X_r) = \sum_{n\in B_r} P(A_n) = \sum_{n\in B_r} \frac{1}{2^{n+1}}.$$

By the definition of the binary expansion,

$$P(X_r) = \sum_{n\in B_r} \frac{1}{2^{n+1}} = r.$$

□

Corollary 3.14. If r is itself 1-random, then r can compute a set of intrinsic density r.

Proof. Use r in place of X in Theorem 3.13. □

In particular, this constructed set cannot compute an r-random set because no r-random set can be computable from r: a set X is r-random by definition if it is random with respect to some representation of μ_r. In particular, this representation cannot compute X, but every representation of μ_r can compute r as shown by Riemann and Slaman [21] Proposition 2.3. Therefore r cannot compute X either.

Almost all $r \in (0, 1)$ are 1-random, so for almost all r we can apply this corollary to obtain an intrinsic density r set which cannot compute an r-random set. From each such example, we may use Theorem 3.5 to generate more examples.

3.3 MWC-Density Compared to Intrinsic Density

The fact that intrinsic density is equivalent to injection stochasticity motivates us to ask the same questions about other notions of stochasticity found in the literature. Much as in the intrinsic density case, much of what is known about MWC and Church-stochastic sets is either a byproduct of facts about randomness, or a separation from randomness as in the work of Ambos-Spies [1] and Wang [28]. We would like to develop an understanding of MWC-density which does not rely on randomness, as we did for intrinsic density in Section 3.2. This approach will allow us to separate

MWC-density and intrinsic density, in the same spirit as computable and Martin-Löf randomness have been separated. We shall see that the `into` and `within` operations behave similarly for MWC and Church-densities as they do for intrinsic density, however other operations are less well behaved.

As per the remark following Definition 1.20, we need to measure $A \triangleleft f(A)$ for all computable monotone selection functions f to check MWC-density, and the total such f to check Church-density. (Recall that $f(A) = \{n : f(A \upharpoonright n) \downarrow = 1\}$.) Throughout this section, we shall focus on MWC-density. However, all of our results will go through for Church-density as well. We will often be given a monotone selection function and need to modify it to suit our needs. Our modification will never make a total monotone selection function not total, so the result will hold in the Church-density case as well.

Much like the intrinsic density case, it is not hard to see that every real in the open unit interval is achieved as the MWC-density of some set, as argued by Bienvenu [Personal Communication]. Pick a set X by putting n in X with probability r. For a fixed total computable monotone selection function f that selects infinitely many bits, after applying f to X we obtain a sequence of independent r-Bernoulli random variables. Then this has asymptotic density r with probability 1. Because there are only countably many total computable monotone selection functions, X has Church-density r with probability 1. This argument relativizes to $\emptyset'$, which implies that the same is true for MWC-density r sets.

From a computability theory perspective, we can make this very explicit. As in the intrinsic density case, sets sufficiently random with respect to μ_r will have MWC-density r. (As in Proposition 1.19, this was previously known and follows from standard arguments. We provide a proof for completeness.)

Proposition 3.15. For $r \in (0, 1)$, if X is r-1-Random, then X has MWC-density r.

Proof. We shall argue by contrapositive. Let f be a monotone selection function, and suppose that $\overline{\rho}(X \triangleleft f(X)) > \epsilon > r$ for some rational ϵ. It is sufficient to consider this case, for if $\underline{\rho}(X \triangleleft f(X)) < r$, then $\overline{\rho}(\overline{X} \triangleleft \overline{f}(\overline{X})) > 1 - r$, where $\overline{f}$ is the monotone selection function obtained by flipping the bits then applying f. Additionally, without loss of generality, we may assume in the partial case that $f(X \upharpoonright n) \downarrow$ for all n. Given f, we know that infinitely often $\rho_n(X \triangleleft f(X)) > \epsilon$. Therefore, whenever we have some σ witnessing this fact by stage s, we may force $f(\tau)$ to converge to 0 for all $\tau \preceq \sigma$ that have not converged by stage s. As in Lemma 1.18, if r is noncomputable, then we are implicitly

working relative to an arbitrary representation for μ_r, which can compute r.

We shall construct an r-(super)martingale that succeeds on X using f. Let α be as in the proof of Proposition 1.19 for r and ϵ, such that $(1+\frac{\alpha}{r})^{\epsilon}(1-\frac{\alpha}{1-r})^{1-\epsilon} > 1$. Define $m : 2^{<\omega} \to \{0,1\}$ as follows:

- $m(\emptyset) = 1$.
- If $f(\sigma) = 1$, let $m(\sigma 1) = (1+\frac{\alpha}{r})m(\sigma)$ and $m(\sigma 0) = (1-\frac{\alpha}{1-r})m(\sigma)$.
- If $f(\sigma) = 0$, let $m(\sigma 1) = m(\sigma 0) = m(\sigma)$.
- If $f(\sigma)\uparrow$, let $m(\sigma 1) = m(\sigma 0) = 0$.

Note that m is a c.e. r-(super)martingale. Furthermore, since $f(X \upharpoonright k)\downarrow$ for all k, $m(X \upharpoonright n) \neq 0$ for all n. Thus,

$$m(X \upharpoonright n) = (1+\frac{\alpha}{r})^{|X\upharpoonright n|}(1-\frac{\alpha}{1-r})^{n-|X\upharpoonright n|}.$$

If $\rho_s(X \triangleleft f(X)) > \epsilon$, then let $n = p_{f(X)}(s)$, the s-th element of $f(X)$. Then

$$|(X \triangleleft f(X)) \upharpoonright s| = |\{k < n : f(X \upharpoonright k) = 1 \text{ and } k \in X\}| \geq \epsilon s,$$

so it follows that

$$m(X \upharpoonright n) \geq (1+\frac{\alpha}{r})^{\epsilon s}(1-\frac{\alpha}{1-r})^{(1-\epsilon)s} = ((1+\frac{\alpha}{r})^{\epsilon}(1-\frac{\alpha}{1-r})^{1-\epsilon})^{s}.$$

By our choice of α, $(1+\frac{\alpha}{r})^{\epsilon}(1-\frac{\alpha}{1-r})^{1-\epsilon} > 1$, so $\sup_{n\to\infty} m(X \upharpoonright n) = \infty$ because there are infinitely many s satisfying the statement above. Thus, m succeeds on X and therefore X is not r-1-Random. □

This shows that every real in the interior of the unit interval is achieved as the MWC-density of a set.[3] There is a body of literature on sets of intrinsic density 0 and 1. There does not seem to be much known about sets of MWC-density 0 or 1. Technically, ω and $\emptyset$ complete the whole unit interval, albeit trivially. However, nontrivial sets of MWC-density 0 should exist simply by being "small enough" to appear small under countably many selection functions. This turns out to be the case.

[3] If f is a total computable selection function, then the above m is a computable r-martingale and therefore this shows that computable randoms have Church-density r. Ambos-Spies [1] exhibited a computable random which is not MWC stochastic, so in general m need not be computable. Wang [28] showed that Schnorr randoms need not be Church stochastic, so the above result does not extend to r-Schnorr randoms for either type of density.

Proposition 3.16. There exists an infinite set $A \leq_T \emptyset'$ such that A has MWC-density (and, therefore, Church-density) 0.

Proof. The construction is similar in principle to the jump strategy for constructing intrinsically small sets from Lemma 2.1. However, the details are more complicated. The reason for this is that monotone selection rules can change their behavior on different inputs, whereas permutations cannot. It is not enough to simply choose sufficiently large elements to enter our set, for a given monotone selection rule may refuse to act until it sees an element enter the set. We utilize the power of the jump to determine if, for a given monotone selection function f, it is possible to force a large gap into $A \triangleleft f(A)$ and ensure that the density is small. If it is not possible, then we do not allow anything into $A \triangleleft f(A)$ until a large gap appears naturally. If no such gap appears, then $A \triangleleft f(A)$ will be finite and we succeed.

Formally, let f_i be an enumeration of the partial computable monotone selection functions. The basic module for ensuring that $\overline{\rho}(A \triangleleft f_i(A)) = 0$ for this specific f_i is as follows: After seeing the n-th 1 enter $A \triangleleft f_i(A)$ at σ_s, we do not allow another 1 to enter until we see n^2 0's enter. (Notice that convergence is not an issue, because the jump can determine if $f_i(\sigma) \downarrow$ uniformly in σ and i.) We will attempt to achieve this by picking some m such that $f_i(\sigma_s 0^k) = 1$ for n^2 k's less than m and setting $\sigma_{s+1} = \sigma_s 0^m 10$. The jump can determine if such an m exists.

Suppose we have defined $\sigma \preceq A$ and there is no m such that $f_i(\sigma 0^m) = 1$. Then we cannot force anything into $A \triangleleft f_i(A)$ without adding extra 1's to A, potentially adding some 1's to $f_j(A)$ for some $j \neq i$. To fix this issue, we say f_i is paused for σ if there does not exist an m such that $f_i(\sigma 0^m) = 1$. As mentioned above, $\emptyset'$ can determine if f_i is paused for σ. When determining how to extend σ_s to $\sigma_{s+1} = \sigma_s 0^m 10$, if $f_i(\sigma_s 0^m 1) = 1$, then σ_{s+1} puts a 0 into $A \triangleleft f_i(A)$. If not, then nothing changes. In both cases, no 1's are added to $A \triangleleft f_i(A)$ by σ_s. We continue and ask if f_i is paused for σ_{s+1}. Either we will eventually see enough 0's enter $A \triangleleft f_i(A)$ after some number of stages and be allowed to add a 1, or this will not happen and $f_i(A)$ will be finite. We succeed in both cases.

We say f_i is almost paused for σ if there is some k such that f_i is paused for $\sigma 0^k$ and $\sigma 0^k$ does not put enough zeroes into $A \triangleleft f_i(A)$. Here, to say f_i is paused means we cannot force another 0 into $A \triangleleft f_i(A)$ by only adding 0's to A. To say f_i is almost paused means we may be able to force some zeroes into $A \triangleleft f_i(A)$, but we cannot force enough zeroes into $A \triangleleft f_i(A)$. (Being almost paused resembles a Σ^0_2 question, but the bound on the number of zeroes necessary reduces it to a question

the jump can answer: we can ask if f_i is paused for σ_s: If so, then it is almost paused. If not, then extend to $\sigma_s 0^k$, where k witnesses that f_i is not paused for σ_s. This adds a zero to $A \triangleleft f_i(A)$. Now ask if f_i is paused for $\sigma_s 0^k$ and repeat. Eventually we will either reach a point where it is paused or we will put enough zeroes into $A \triangleleft f_i(A)$.)

Finally, we describe the construction using this module on all i simultaneously: at stage s, we consider only the $i \leq s$. Using the jump, determine which i are almost paused at stage s and ignore them. For the remaining i, we may choose m large enough such that $\sigma_{s+1} = \sigma_s 0^m 10$ puts enough zeroes into $A \triangleleft f_i(A)$ to ensure n^2 zeroes are enumerated before the $(n+1)$-st 1, where n is the current number of 1's in $A \triangleleft f_i(A)$. As we are ignoring all of the almost paused selection functions, we can always extend to σ_{s+1} and thus A is infinite. Furthermore, $\overline{\rho}(A \triangleleft f_i(A)) = 0$ for all i since either $f_i(A)$ is finite or $\rho_n(A \triangleleft f_i(A)) \leq \frac{k+1}{k^2+k+1}$ for increasing k. (If there are $k+1$ ones in $|A \triangleleft f_i(A) \upharpoonright n|$, then there are at least k^2 zeroes between the final two.) □

This still leaves open MWC-density 1, for it may not be obvious a priori that MWC-density behaves as intrinsic density does with respect to complements. We can prove that the behavior is the same, however. This gives a non-trivial example of an MWC-density 1 set.

Proposition 3.17. Let A have MWC-density α. Then $\overline{A}$ has MWC-density $1 - \alpha$.

Proof. Let f be a computable monotone selection function. Define $\overline{f} : 2^{<\omega} \to \{0,1\}$ via $\overline{f}(\sigma) = f(1-\sigma)$, where $1 - \sigma = \tau \in 2^{|\sigma|}$ with $\tau(n) = 1 - \sigma(n)$ for all $n < |\sigma|$. Then $f(\overline{A} \upharpoonright n) = \overline{f}(A \upharpoonright n)$. Since A has MWC-density α and $\overline{f}$ is a computable monotone selection function, either $\overline{f}(A)$ is finite or $\rho(A \triangleleft \overline{f}(A)) = \alpha$. If the former, then $f(\overline{A})$ is also finite. If the latter, then

$$\overline{A} \triangleleft f(\overline{A}) = \overline{A} \triangleleft \overline{f}(A) = \overline{A \triangleleft \overline{f}(A)}.$$

Therefore,

$$\rho(\overline{A} \triangleleft f(\overline{A})) = \rho(\overline{A \triangleleft \overline{f}(A)}) = 1 - \rho(A \triangleleft \overline{f}(A)) = 1 - \alpha,$$

as desired. □

Having obtained the whole unit interval in nontrivial fashion, we now turn to investigating MWC-density analogs of results from Section 1.20 to illustrate the differences between the two notions of stochasticity. We begin with `into` and `within`, which behave in nearly the same fashion.

Theorem 3.18. Suppose C is computable and A has MWC-density α. Then $A \triangleleft C$ has MWC-density α.

Proof. Let f be a computable monotone selection function. Define $\hat{C} : 2^{<\omega} \to 2^{<\omega}$ via $\hat{C}(\sigma) = \tau$ with $\tau \in 2^{max(n:c_n<|\sigma|)+1}$ and $\tau(i) = \sigma(c_i)$ for all $i < |\tau|$. Notice that $\hat{C}(X \upharpoonright c_n) = (X \triangleleft C) \upharpoonright n$ by definition.

Now, define $f_C : 2^{<\omega} \to \{0,1\}$ via $f_C(\sigma) = 1$ if and only if $|\sigma| = c_i$ for some i and $f(\hat{C}(\sigma)) = 1$. Since C is computable, $\hat{C}$ is computable and, thus, f_C is a computable monotone selection function. We now show that $A \triangleleft f_C(A) = (A \triangleleft C) \triangleleft f(A \triangleleft C)$.

We shall show that $(A \triangleleft C) \triangleleft f(A \triangleleft C) \subseteq A \triangleleft f_C(A)$ with a sequence of equivalent statements, therefore proving the reverse as well. A number n is in $(A \triangleleft C) \triangleleft f(A \triangleleft C)$ if and only if the n-th element of $f(A \triangleleft C)$ is in $A \triangleleft C$, i.e. the n-th k with $f((A \triangleleft C) \upharpoonright k) = 1$ is in $A \triangleleft C$. This occurs if and only if $c_k \in A$. Now, note that $f_C(A)$ is the set of all c_i such that $f(\hat{C}(A \upharpoonright c_i)) = f((A \triangleleft C) \upharpoonright i) = 1$, so k is as above if and only if $c_k \in f_C(A)$ and $c_k \in A$. Note that c_k must be the n-th element of $f_C(A)$ because k was the n-th number with $f((A \triangleleft C) \upharpoonright k) = 1$, so $n \in A \triangleleft f_C(A)$.

Since A has MWC-density α,

$$\rho((A \triangleleft C) \triangleleft f(A \triangleleft C)) = \rho(A \triangleleft f_C(A)) = \alpha.$$

Since f was arbitrary, $A \triangleleft C$ also has MWC-density α. □

To prove the analog of Theorem 3.9 for MWC-density, we require more relativization. We shall see that this is the main theme with MWC-density compared to intrinsic density. Unlike intrinsic density, where the selection and interpretation functions act independently of the input set, MWC-density can change the selected bits based on finitely much of the input set. This means that if B is related to A in some predictable fashion, then a monotone selection rule may be able to use information from B to predict bits of A. Assuming the sets have MWC-density relative to each other will avoid this issue since using B as an oracle will allow us to simulate an input set involving B, and vice versa for A. We shall see some consequences of this distinction after Theorem 3.21. To prove this theorem, however, we shall require the following technical observation. The proof involves merely unraveling definitions, but we provide it for clarity, because the definitions can be cumbersome.

Lemma 3.19. Let A, B and C be sets. Then

$$(A \triangleleft C) \triangleleft (B \triangleleft C) = A \triangleleft (B \cap C).$$

Proof. By definition,

$$A \triangleleft (B \cap C) = \{n : p_{B \cap C}(n) \in A\}.$$

That is, it is the set consists of those n such that the n-th element of $B \cap C$ is in A.

Similarly, by definition,

$$(A \triangleleft C) \triangleleft (B \triangleleft C) = \{n : p_{B \triangleleft C}(n) \in A \triangleleft C\}.$$

That is, the set consists of those n such that the n-th element of $B \triangleleft C$ is in $A \triangleleft C$. However, if $k \in A \triangleleft C$ for some k, this means $c_k \in A$, by definition. Therefore, if $n \in (A \triangleleft C) \triangleleft (B \triangleleft C)$, this translates to $c_{p_{B \triangleleft C}(n)} \in A$. Since $p_{B \triangleleft C}(n)$ is the n-th element of $B \triangleleft C$, $c_{p_{B \triangleleft C}(n)}$ is the n-th element of C which is in B. That is, $c_{p_{B \triangleleft C}(n)}$ is the n-th element of $B \cap C$. This confirms that the sets are identical. □

Here is a corollary of this lemma which we will not need, but does improve our calculus of `into` and `within`.

Corollary 3.20. For any sets A, B, and C,

$$(A \triangleleft C) \triangleleft (B \triangleleft C) = (A \triangleleft B) \triangleleft (C \triangleleft B).$$

Proof. As intersection is symmetric,

$$A \triangleleft (B \cap C) = A \triangleleft (C \cap B).$$

Therefore, applying Lemma 3.19 once on each side tells us that

$$(A \triangleleft C) \triangleleft (B \triangleleft C) = (A \triangleleft B) \triangleleft (C \triangleleft B).$$

□

Now we are ready to prove the analog of Theorem 3.9.

Theorem 3.21. Suppose that A has MWC-density α relative to B and B has MWC-density β relative to A. Then $B \triangleright A$ has MWC-density $\alpha\beta$.

Proof. The proof is similar to the proof of Theorem 3.9. However, there is an extra consideration for MWC-density because the selected bits can depend on the input. In Theorem 3.9, $\pi(B \triangleright A)$ is a subset of $\pi(A)$, so we send B to $\pi(B \triangleright A) \triangleleft \pi(A)$ (modulo a set of density zero) and apply Lemma 3.7. However, we don't know in general whether $A \triangleleft f(A)$ contains $(B \triangleright A) \triangleleft f(B \triangleright A)$ because $f(B \triangleright A)$ need not be a subset of $f(A)$. We first construct a B-computable monotone selection function f_B such that $f_B(A) = f(B \triangleright A)$ and, therefore, $A \triangleleft f_B(A)$ is a superset of $(B \triangleright A) \triangleleft f(B \triangleright A)$. Then because A has MWC-density α relative to B, $A \triangleleft f_B(A)$ will have density α. At this point, we shall borrow the proof idea of Theorem 3.9; namely, we shall construct an A-computable monotone selection function f_A such that

$$B \triangleleft f_A(B) = ((B \triangleright A) \triangleleft f(B \triangleright A)) \triangleleft (A \triangleleft f_B(A)).$$

Because B has MWC-density β with respect to A, $B \triangleleft f_A(B)$ will have density β. We may then apply Lemma 3.7 to show that $(B \triangleright A) \triangleleft f(B \triangleright A)$ has density $\alpha\beta$, as desired.

Formally, let f be a computable monotone selection function. If $f(B \triangleright A)$ is finite or undefined, we are done. If not, define $f_B : 2^{<\omega} \to \{0, 1\}$ via $f_B(\sigma) = f(B \triangleright \sigma)$, where $B \triangleright \sigma \in 2^{|\sigma|}$ is defined as one might expect: $B \triangleright \sigma(n) = 1$ if and only if $\sigma(n) = 1$ and n is the b_i'th m such that $\sigma(m) = 1$ for some $i \in \omega$. Since $(X \triangleright Y) \upharpoonright n = X \triangleright (Y \upharpoonright n)$, it is immediate that

$$f_B(A) = \{n : f_B(A \upharpoonright n) = 1\} = \{n : f(B \triangleright (A \upharpoonright n)) = 1\} =$$

$$\{n : f((B \triangleright A) \upharpoonright n) = 1\} = f(B \triangleright A).$$

Therefore, since $B \triangleright A \subseteq A$,

$$(B \triangleright A) \triangleleft f(B \triangleright A) = (B \triangleright A) \triangleleft f_B(A) \subseteq A \triangleleft f_B(A).$$

Let

$$X = ((B \triangleright A) \triangleleft f(B \triangleright A)) \triangleleft (A \triangleleft f(B \triangleright A)).$$

We shall construct an A-computable monotone selection function f_A such that $B \triangleleft f_A(B) = X$ via Lemma 3.19.

Let $f_A : 2^{<\omega} \to \{0,1\}$ be defined via $f_A(\sigma) = f(\sigma \triangleright A)$, where $\sigma \triangleright A = \tau \in 2^{a_{|\sigma|}}$ is defined via $\tau(n) = 1$ if and only if $n = a_m$ and $\sigma(m) = 1$ for some $m < |\sigma|$. We now claim that $B \triangleleft f_A(B) = (B \triangleright A) \triangleleft (A \cap f(B \triangleright A))$.

If $n \in (B \triangleright A) \triangleleft (A \cap f(B \triangleright A))$, then the n-th element of $A \cap f(B \triangleright A)$ is in $B \triangleright A$ by the definition of the `within` operation. This implies that n is of the form a_m for $m \in B$, where m is the n-th number k such that $a_k \in A \cap f(B \triangleright A)$. Since a_m is in $f(B \triangleright A)$, by the definition of f_A, this implies that m is the n-th number with

$$f((B \triangleright A) \upharpoonright a_m) = f((B \upharpoonright m) \triangleright A) = f_A(B \upharpoonright m) = 1.$$

Thus, m is the n-th element of $f_A(B)$, and it lies in B, so $m = p_{f_A(B)}(n) \in B$. Therefore, $n \in B \triangleleft f_A(B)$. As n was arbitrary,

$$(B \triangleleft A) \triangleleft (A \cap f(B \triangleright A)) \subseteq B \triangleleft f_A(B).$$

This argument reverses, so $B \triangleleft f_A(B) = (B \triangleright A) \triangleleft (A \cap f(B \triangleright A))$.

Therefore,

$$X = ((B \triangleright A) \triangleleft f(B \triangleright A)) \triangleleft (A \triangleleft f(B \triangleright A)) = (B \triangleright A) \triangleleft (A \cap f(B \triangleright A)) = B \triangleleft f_A(B).$$

The first equality is by definition, the second is by Lemma 3.19, and the final is from the previous paragraph. This implies that

$$X \triangleright (A \triangleleft f_B(A)) = (B \triangleleft f_A(B)) \triangleright (A \triangleleft f_B(A)).$$

Since A has MWC-density α with respect to B and $f_B(A) = f(B \triangleright A)$, we have $\rho(A \triangleleft f(B \triangleright A)) = \alpha$. Since B has MWC-density β with respect to A, $\rho(B \triangleleft f_A(B)) = \beta$. Therefore, by Lemma 3.7,

$$\rho((B \triangleleft f_A(B)) \triangleright (A \triangleleft f(B \triangleright A))) = \rho(B \triangleleft f_A(B))\rho(A \triangleleft f(B \triangleright A)) = \alpha\beta.$$

Finally, recall from the definition of X that $X \triangleright (A \triangleleft f(B \triangleright A)) = (B \triangleright A) \triangleleft f(B \triangleright A)$. Therefore,

$$\rho((B \triangleright A) \triangleleft f(B \triangleright A)) = \rho(X \triangleright (A \triangleleft f(B \triangleright A))) = \alpha\beta,$$

as desired. □

The extra relativization used in Theorem 3.21 rears its head immediately. Following Theorem 3.9, we were able to obtain as an easy corollary that if A has intrinsic density α and B has intrinsic density β relative to A, then $A \cap B$ has intrinsic density $\alpha\beta$. The proof simply observed that $B \triangleleft A$ had intrinsic density β relative to A via the relativized form of Theorem 3.5 and then applied Theorem 3.9 because $(B \triangleleft A) \triangleright A = A \cap B$. In the MWC-density case, this argument fails. Theorem 3.21 requires relativization in both directions, and while the relativized form of Theorem 3.18 ensures that $B \triangleleft A$ has MWC-density β relative to A, it does not ensure that A has MWC-density α relative to $B \triangleleft A$, so we cannot apply Theorem 3.21 as we wish. Whether this relativization is necessary remains an open question, which we shall state fully in Question 4.9.

Fortunately, we can recover the intersection property for relatively MWC-dense sets using an alternate proof, so this is still not a major departure from what is true for intrinsic density.

Proposition 3.22. If A has MWC-density α relative to B and B has MWC-density β relative to A, then $A \cap B$ has MWC-density $\alpha\beta$.

Proof. Let f be a computable monotone selection function. If $f(A \cap B)$ is finite, then we are done. Otherwise, consider $(A \cap B) \triangleleft f(A \cap B)$. Define the B-computable monotone selection function $f_B : 2^{<\omega} \to \{0,1\}$ via $f_B(\sigma) = 1$ if and only if $f(\sigma \cap B) = 1$, where $\sigma \cap B = \tau \in 2^{|\sigma|}$ is given by $\tau(n) = 1$ if and only if $\sigma(n) = 1$ and $B(n) = 1$. Then, clearly, $f_B(A) = f(A \cap B)$, so $A \triangleleft f_B(A) = A \triangleleft f(A \cap B)$. Since A has MWC-density α relative to B,

$$\rho(A \triangleleft f_B(A)) = \rho(A \triangleleft f(A \cap B)) = \alpha.$$

We shall now construct an A-computable monotone selection function f_A such that

$$B \triangleleft f_A(B) = (B \triangleleft f(A \cap B)) \triangleleft (A \triangleleft f(A \cap B))$$

via Lemma 3.19.

Define $f_A : 2^{<\omega} \to \{0,1\}$ via $f_A(\sigma) = 1$ if and only if $f(A \cap \sigma) = 1$ and $|\sigma| \in A$, where $A \cap \sigma$ is defined similarly to $\sigma \cap B$ in the obvious way. Then it follows immediately that $f_A(B) = A \cap f(A \cap B)$, so

$$B \lhd f_A(B) = B \lhd (A \cap f(A \cap B).$$

By Lemma 3.19,

$$B \lhd (A \cap f(A \cap B) = (B \lhd f(A \cap B)) \lhd (A \lhd f(A \cap B)).$$

Therefore, by the properties of the `within` operation, we have that

$$(B \lhd f_A(B)) \rhd (A \lhd f_B(A)) = (B \lhd f(A \cap B)) \lhd (A \lhd f(A \cap B)) \rhd (A \lhd f(A \cap B)) =$$

$$(B \lhd f(A \cap B)) \cap (A \lhd f(A \cap B)) = (A \cap B) \lhd f(A \cap B).$$

By Lemma 3.7,

$$\rho((B \lhd f_A(B)) \rhd (A \lhd f_B(A)) = \rho((A \cap B) \lhd f(A \cap B)) = \alpha\beta.$$

Since f was arbitrary, $A \cap B$ has MWC-density $\alpha\beta$. □

So far, we have seen similar behavior between intrinsic density and MWC-density. This trend will not continue, however. Both the join and the union are more volatile in the setting of MWC-density, but we will be able to recover some utility.

Where Theorem 3.3 says that in a specific sense intrinsic density is ignorant of (computable) internal structure, the opposite is true of MWC-density. In fact, the analog of Theorem 3.3 for MWC-density fails in very strong fashion.

Proposition 3.23. Suppose that A has MWC-density α for $0 \leq \alpha < 1$. Then $A \oplus A$ does not have MWC-density.

Proof. Let E be the set of even numbers and let O be the set of odd numbers. Define $f : 2^{<\omega} \to \{0,1\}$ via $f(\sigma) = 1$ if $|\sigma| \in O$ and $\sigma(|\sigma|-1) = 1$ and $f(\sigma) = 0$ otherwise. Then for any A, $f(A \oplus A) = A \rhd O$. Therefore,

$$(A \oplus A) \lhd f(A \oplus A) = (A \oplus A) \lhd (A \rhd O) = \omega,$$

so

$$\rho((A \oplus A) \lhd f(A \oplus A)) = 1.$$

However, because A has MWC-density $\alpha < 1$, it has density α and $A \oplus A$ has density α. Therefore, $A \oplus A$ cannot have MWC-density, since its asymptotic density does not match the density of $(A \oplus A) \triangleleft f(A \oplus A)$. □

It is possible to show that the join preserves MWC-density if both sets have the same MWC-density relative to one another, as one might guess given Van Lambaglen's theorem for randomness.

Theorem 3.24. If A has MWC-density r relative to B and B has MWC-density r relative to A, then $A \oplus B$ has MWC-density r.

Proof. Let E be the set of even numbers and let O be the set of odd numbers. Given a computable monotone selection function f, if $f(A \oplus B)$ is finite then we are done. Therefore assume it is infinite. We want to prove that

$$\rho((A \oplus B) \triangleleft f(A \oplus B)) = r.$$

Define the B-computable monotone selection function f_B via $f_B(\sigma) = f((\sigma \oplus B) \upharpoonright 2|\sigma|)$. Notice that $f_B(A) = f(A \oplus B) \triangleleft E$: $n \in f_B(A)$ if and only if

$$f_B(A \upharpoonright n) \downarrow = f(((A \upharpoonright n) \oplus B) \upharpoonright 2n) = f(A \oplus B) \upharpoonright 2n) \downarrow = 1.$$

Therefore, $2n \in f(A \oplus B)$ if and only if $n \in f_B(A)$. Thus, $f(A \oplus B) \triangleleft E = f_B(A)$. We shall now use the following lemma:

Lemma 3.24.1. Given X and Y, $(X \triangleright Y) \triangleleft Y = X$.

Proof. By definition,

$$(X \triangleright Y) \triangleleft Y = \{n : y_n \in (X \triangleright Y)\} = \{n : y_n = y_{x_k} \text{ for some } k \in \omega\} =$$

$$\{n : n = x_k \text{ for some } k \in \omega\} = X.$$

□

Therefore, by Lemma 3.24.1, we can rewrite A as $(A \triangleright E) \triangleleft E$. Thus, we have

$$A \triangleleft f_B(A) = A \triangleleft (f(A \oplus B) \triangleleft E) = ((A \triangleright E) \triangleleft E) \triangleleft (f(A \oplus B) \triangleleft E).$$

Applying Lemma 3.19, we get

$$A \triangleleft f_B(A) = ((A \triangleright E) \triangleleft E) \triangleleft (f(A \oplus B) \triangleleft E) = (A \triangleright E) \triangleleft (f(A \oplus B) \cap E).$$

Since A has MWC-density r relative to B, $\rho((A \triangleright E) \triangleleft (f(A \oplus B) \cap E)) = \rho(A \triangleleft f_B(A)) = r$.

A similar argument with f_A defined similarly to f_B, replacing the roles of the evens with the odds and the role of A with B, shows that $\rho((B \triangleright O) \triangleleft (f(A \oplus B) \cap O)) = \rho(B \triangleleft f_A(B)) = r$ since B has MWC-density r relative to A.

If $f(A \oplus B) \cap O$ is finite, then

$$(A \oplus B) \triangleleft f(A \oplus B) = ((A \triangleright E) \triangleleft f(A \oplus B)) \sqcup ((B \triangleright O) \triangleleft f(A \oplus B))$$

will differ from $(A \triangleright E) \triangleleft (f(A \oplus B) \cap E)$ by finitely much. It will therefore also have asymptotic density r as desired. By a symmetric argument, the same applies if $f(A \oplus B) \cap E$ is finite. Therefore, we can assume that both are infinite, and it then suffices to prove the following lemma:

Lemma 3.24.2. Suppose $X \cap H$ is infinite and coinfinite. Then if $\rho((A \triangleright H) \triangleleft (X \cap H)) = r$ and $\rho((B \triangleright \overline{H}) \triangleleft (X \cap \overline{H})) = r$, we have $\rho((A \oplus_H B) \triangleleft X) = r$.

Proof. By the definition of asymptotic density and the within operation,

$$\rho((A \oplus_H B) \triangleleft X) = \lim_{n \to \infty} \frac{|\{k < p_X(n) : k \in X \cap (A \oplus_H B)\}|}{n}$$

We can use the fact that $(A \oplus_H B) \cap X \cap H = (A \triangleright H) \cap X$ and $(A \oplus_H B) \cap X \cap \overline{H} = (B \triangleright \overline{H}) \cap X$ to split the numerator and obtain

$$\lim_{n \to \infty} \frac{|\{k < p_X(n) : k \in X \cap (A \triangleright H)\}|}{n} + \frac{|\{k < p_X(n) : k \in X \cap (B \triangleright \overline{H})\}|}{n}.$$

Let $t = |(X \cap H) \upharpoonright p_X(n)|$, that is, t is the number of elements of $X \cap H$ in the first n elements of X. Then $n - t$ will be the number of elements of $X \cap \overline{H}$ in the first n elements of X. In particular,

$$|\{k < p_X(n) : k \in X \cap (A \triangleright H)\}| = |\{k < p_{X \cap H}(t) : k \in X \cap (A \triangleright H)\}|$$

and

$$|\{k < p_X(n) : k \in X \cap (B \triangleright \overline{H})\}| = |\{k < p_{X \cap \overline{H}}(n-t) : k \in X \cap (B \triangleright \overline{H})\}|.$$

These equalities can be used to rearrange the above limit and obtain

$$\lim_{n\to\infty} \frac{|\{k < p_{X\cap H}(t) : k \in X \cap (A \triangleright H)\}|}{t} \cdot \frac{t}{n} + \frac{|\{k < p_{X\cap \overline{H}}(n-t) : k \in X \cap (B \triangleright \overline{H})\}|}{n-t} \cdot \frac{n-t}{n}.$$

Notice that

$$\lim_{t\to\infty} \frac{|\{k < p_{X\cap H}(t) : k \in X \cap (A \triangleright H)\}|}{t} = \rho((A \triangleright H) \triangleleft (X \cap H)) = r$$

and

$$\lim_{n-t\to\infty} \frac{|\{k < p_{X\cap \overline{H}}(n-t) : k \in X \cap (B \triangleright \overline{H})\}|}{n-t} = \rho((B \triangleright \overline{H}) \triangleleft X \cap \overline{H}) = r.$$

Because both $X \cap H$ and $X \cap \overline{H}$ are infinite, both t and $n-t$ go to infinity as n goes to infinity. Therefore, as n goes to infinity, the first fraction in each term goes to r and the limit becomes

$$\rho((A \oplus_H B) \triangleleft X) = \lim_{n\to\infty} r \cdot \frac{t}{n} + r \cdot \frac{n-t}{n} = r \cdot \lim_{n\to\infty} \frac{nt + n^2 - nt}{n^2} = r \cdot 1 = r,$$

as desired. □

By Lemma 3.24.2, with E playing the role of H and $f(A \oplus B)$ playing the role of X, $\rho(A \oplus B) \triangleleft f(A \oplus B)) = r$, as desired. □

However, Proposition 3.23 leaves us unable to generalize the proof that relativization was necessary in the statement of Theorem 3.9 to the analogous proof for Theorem 3.21. This illustrates a critical difference between intrinsic density and MWC-density. It also blocks us from studying the Turing degrees of MWC-dense sets in the same fashion as we did in Section 2.4, as our techniques relied heavily on the fact that the join preserved intrinsic density without relativization.

Not only does the join fail to behave well for MWC-density, but we shall in fact see that the union also does not behave well. The difficulty lies in the fact that the bits selected by f on $A \sqcup B$ need not, in general, be the union of the bits selected by f on A and the bits selected by f on B. On one hand, it is not difficult to prove that if A has MWC-density α relative to B and B has MWC-density β relative to A with A and B disjoint, then $A \sqcup B$ has MWC-density $\alpha + \beta$. Given a monotone selection function f, there is a B-computable monotone selection function f_B such that

$f_B(A) = f(A \sqcup B)$. Similarly, there is an A-computable monotone selection function f_A such that $f_A(B) = f(A \sqcup B)$. Then, by the properties of the `within` operation,

$$(A \sqcup B) \triangleleft f(A \sqcup B) = (A \triangleleft f(A \sqcup B)) \sqcup (B \triangleleft f(A \sqcup B)) = (A \triangleleft f_B(A)) \sqcup (B \triangleleft f_A(B)).$$

Therefore,

$$\rho((A \sqcup B) \triangleleft f(A \sqcup B)) = \rho(A \triangleleft f_B(A)) + \rho(B \triangleleft f_A(B)) = \alpha + \beta.$$

However, Proposition 3.22 ensures that $A \cap B = \emptyset$ implies that one of A or B has MWC-density 0 under these assumptions, so this result cannot be used to obtain new MWC-densities as the disjoint unions of sets that have intrinsic density relative to one another.

One may think to drop the requirements that A and B have MWC-density relative to one another, therefore disallowing the use of Proposition 3.22 and avoiding this problem. However, the union still need not have MWC-density. The following lemma will allow us to construct such an example.

Lemma 3.25. If A has MWC-density 0 and g is an increasing, total, computable function, then $B = \{a_n + g(n) : n \in \omega\}$ also has MWC-density 0.

Proof. We argue by contrapositive. Let f be a monotone selection function such that $\overline{\rho}(B \triangleleft f(B)) > 0$. We shall construct a monotone selection function $\hat{f}$ such that $\overline{\rho}(A \triangleleft \hat{f}(A)) \geq \overline{\rho}(B \triangleleft f(B)) > 0$.

Given $\sigma \in 2^{<\omega}$, let $\sigma_0 < \sigma_1 < \cdots < \sigma_k$ represent all indices on which σ is 1. Define $g(\sigma)$ to be $\tau \in 2^{|\sigma|+g(k+1)}$ with $\tau(i) = 1$ if and only if $i = \sigma_j + g(j)$ for some $j \leq k$. Finally, define $\hat{f} : 2^{<\omega} \to \{0,1\}$ via $\hat{f}(\sigma) = 1$ if and only if $f(g(\sigma)) = 1$. Suppose that $n \in B \triangleleft f(B)$. Then $p_{f(B)}(n) = a_k + g(k)$ for some $k \in \omega$. In particular, $f(B \upharpoonright a_k + g(k)) = 1$. Therefore, since $g(A \upharpoonright a_k) = B \upharpoonright a_k + g(k)$, by the definition of $g(\sigma)$, we have

$$\hat{f}(A \upharpoonright a_k) = f(g(A \upharpoonright a_k)) = f(B \upharpoonright a_k + g(k)) = 1.$$

Finally, notice that the m such that $p_{\hat{f}(A)}(m) = a_k$ must be less than or equal to n, because each element of $\hat{f}(A)$ corresponds to an element of $f(B)$ but not necessarily vice versa. It follows that $\overline{\rho}(A \triangleleft \hat{f}(A)) \geq \overline{\rho}(B \triangleleft f(B))$, since each element of $B \triangleleft f(B)$ corresponds to an element of $A \triangleleft \hat{f}(A)$ which is no larger. Since $\overline{\rho}(B \triangleleft f(B)) > 0$, we are done. □

Proposition 3.26. There exists a set A such that A and $A \triangleright \overline{A}$ both have MWC-density, but $A \sqcup (A \triangleright \overline{A})$ does not.

Proof. By Proposition 3.16, there is an infinite set X with MWC-density 0. By Lemma 3.25, $A = \{x_n + n^2 : n \in \omega\}$ also has MWC-density 0. Notice that

$$a_{n+1} - a_n = x_{n+1} + (n+1)^2 - x_n - n^2 = x_{n+1} - x_n + 2n + 1 > 2n + 1.$$

It follows that the a_n-th element of $\overline{A}$ is a_n+n+1. (The only way this could fail is if $a_{n+1} \leq a_n+n+1$.) Therefore $A \triangleright \overline{A} = \{a_n + n + 1 : n \in \omega\}$, so it has MWC-density 0 by Lemma 3.25.

Let $f : 2^{<\omega} \to \{0, 1\}$ be defined via $f(\sigma) = 1$ if and only if $m < |\sigma|$ is the largest number with $\sigma(m) = 1$, σ has $2k+1$ 1's, and $|\sigma| = m+k+1$. It is immediate that f is a total monotone selection function, and furthermore $f(A \sqcup (A \triangleright \overline{A})) = A \triangleright \overline{A}$: by the above, $A \sqcup (A \triangleright \overline{A})$ alternates between elements of A and elements of $A \triangleright \overline{A}$. The elements of A signal where elements of $A \triangleright \overline{A}$ will sit, allowing f to select exactly those elements. Therefore,

$$(A \sqcup (A \triangleright \overline{A})) \triangleleft f(A \sqcup (A \triangleright \overline{A})) = (A \sqcup (A \triangleright \overline{A})) \triangleleft (A \triangleright \overline{A}) = \omega.$$

Thus, $A \sqcup (A \triangleright \overline{A})$ does not have MWC-density 0. However, it has density 0 as the union of two sets of density 0, so it does not have MWC-density. □

This shows that disjoint unions in general need not sum MWC-densities, a critical difference between intrinsic density and MWC-density. However, this example relies nontrivially on the fact that the sets have density 0. Is it possible to find an example with sets of positive MWC-density? It turns out that the answer is yes. Bienvenu [Personal Communication] shared the following argument: We shall construct disjoint A and B with both having MWC-density $\frac{1}{2}$ but $A \sqcup B$ does not have MWC-density 1. With probability $\frac{1}{n}$, keep both $2n$ and $2n + 1$ out of both A and B. For all naturals m not explicitly excluded, with independent probability $\frac{1}{2}$ put m into A and put it into B if it does not enter A. Then with probability 1, both A and B will have MWC-density $\frac{1}{2}$. However, $A \sqcup B$ does not have MWC-density 1 because $A \sqcup B$ will be a sequence of 00's and 11's, so whenever we see a 0, we select the next bit and obtain an infinite sequence of 0's. (This will be infinite by the effective version of the second Borel-Cantelli lemma.)

Another potential solution to the problem of misbehaving unions is to remove the requirement

that the sets be disjoint. If A has MWC-density α relative to B and B has MWC-density β relative to A, then must $A \cup B$ have MWC-density $\alpha + \beta - \alpha\beta$? (The inclusion-exclusion principle implies that $\rho_n(A \cup B) = \rho_n(A) + \rho_n(B) - \rho_n(A \cap B)$. Together with Proposition 3.22, this suggests that the MWC-density of $A \cup B$ must be $\alpha + \beta - \alpha\beta$ if it has MWC-density at all.) It turns out that this is true.

Theorem 3.27. Suppose A has MWC-density α relative to B and B has MWC-density β relative to A. Then $A \cup B$ has MWC-density $\alpha + \beta - \alpha\beta$.

Proof. Let f be a computable monotone selection function. If $f(A \cup B)$ is finite, we are done. Otherwise, consider $(A \cup B) \triangleleft f(A \cup B)$. By definition,

$$\rho((A \cup B) \triangleleft f(A \cup B)) = \lim_{n \to \infty} \rho_n((A \cup B) \triangleleft f(A \cup B)).$$

By the inclusion-exclusion principle and the properties of the `within` operation,

$$\rho_n((A \cup B) \triangleleft f(A \cup B)) = \rho_n(A \triangleleft f(A \cup B)) + \rho_n(B \triangleleft f(A \cup B)) - \rho_n((A \cap B) \triangleleft f(A \cup B)).$$

Let $f_A : 2^{<\omega} \to \{0, 1\}$ be defined via $f(\sigma) = 1$ if and only if $f(\sigma \cup A) = 1$, where $\sigma \cup A = \tau \in 2^{|\sigma|}$ with $\tau(n) = 1$ if and only if $\sigma(n) = 1$ or $A(n) = 1$. Let f_B be defined similarly for B in place of A.

Since A has MWC-density α relative to B and B has MWC-density β relative to A,

$$\rho(A \triangleleft f_B(A)) = \rho(A \triangleleft f(A \cup B)) = \lim_{n \to \infty} \rho_n(A \triangleleft f(A \cup B)) = \alpha$$

and

$$\rho(B \triangleleft f_A(B)) = \rho(B \triangleleft f(A \cup B)) = \lim_{n \to \infty} \rho_n(B \triangleleft f(A \cup B)) = \beta.$$

Therefore, what remains is to use an argument similar to that for Proposition 3.22 to handle the intersection.

Define $\hat{f_A} : 2^{<\omega} \to \{0, 1\}$ via $\hat{f_A}(\sigma) = 1$ if and only if $f_A(\sigma) = 1$ and $|\sigma| \in A$. Then it follows immediately that

$$\hat{f_A}(B) = A \cap f_A(B) = A \cap f(A \cup B),$$

so

$$B \triangleleft \hat{f_A}(B) = B \triangleleft (A \cap f(A \cup B).$$

By Lemma 3.19,

$$B \triangleleft (A \cap f(A \cup B) = (B \triangleleft f(A \cup B)) \triangleleft (A \triangleleft f(A \cup B)).$$

Therefore, by the same argument as in Proposition 3.22, we have

$$\rho((B \triangleleft f_A(B)) \triangleright (A \triangleleft f_B(A)) = \rho((A \cap B) \triangleleft f(A \cup B)) = \alpha\beta.$$

Thus, we have $\lim_{n\to\infty} \rho_n((A \cap B) \triangleleft f(A \cup B)) = \alpha\beta$, and it follows that

$$\rho((A \cup B) \triangleleft f(A \cup B)) =$$

$$\lim_{n\to\infty} \rho_n(A \triangleleft f(A \cup B)) + \lim_{n\to\infty} \rho_n(B \triangleleft f(A \cup B)) - \lim_{n\to\infty} \rho_n((A \cap B) \triangleleft f(A \cup B)) = \alpha + \beta - \alpha\beta,$$

as desired. Since f was arbitrary, $A \cup B$ has MWC-density $\alpha + \beta - \alpha\beta$. □

While this will allow us to change MWC-density, the fact that the union is not disjoint can make this difficult to control if we want to iterate. Fortunately, in addition to the general union, we can show that a specific type of disjoint union combines MWC-densities using the same formula. The format and disjointness of this special form is more useful for our attempts to translate the proof of Theorem 3.13 to MWC-density. Unfortunately, we will not completely succeed. We shall discuss the attempts to translate this proof into the MWC-density case and why they only partially succeed below, but we first introduce our special form of disjoint union.

Lemma 3.28. Suppose that A has MWC-density α relative to B and B has MWC-density β relative to A. Then $A \sqcup (B \triangleright \overline{A})$ has MWC-density $\alpha + \beta(1 - \alpha) = \alpha + \beta - \alpha\beta$.

Proof. Let f be a monotone selection function. We wish to show that

$$\rho((A \sqcup (B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \alpha + \beta - \alpha\beta.$$

By the properties of the `within` operation,

$$(A \sqcup (B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A})) = (A \triangleleft f(A \sqcup (B \triangleright \overline{A}))) \sqcup ((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))),$$

so

$$\rho((A \sqcup (B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \rho((A \triangleleft f(A \sqcup (B \triangleright \overline{A})))) + \rho(((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A})))).$$

Therefore, we shall first construct a B-computable monotone selection function f_B such that $f_B(A) = f(A \sqcup (B \triangleright \overline{A}))$. Then

$$A \triangleleft f(A \sqcup (B \triangleright \overline{A})) = A \triangleleft f_B(A),$$

and, therefore, because A has MWC-density α with respect to B, we have

$$\rho(A \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \rho(A \triangleleft f_B(A)) = \alpha.$$

Define $f_B : 2^{<\omega} \to \{0, 1\}$ via $f_B(\sigma) = 1$ if and only if $f(\sigma \sqcup (B \triangleright \overline{\sigma})) = 1$, where $\sigma \sqcup (B \triangleright \overline{\sigma})$ is defined to be $\tau \in 2^{|\sigma|}$ with $\tau(k) = 1$ if and only if $\sigma(k) = 1$ or k is the b_i-th 0 in σ for some i. From this definition, it is immediate that $f_B(A) = f(A \sqcup (B \triangleright \overline{A}))$, as desired.

It remains to be shown that

$$\rho((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \beta(1 - \alpha) = \beta - \beta\alpha.$$

We would like to use Theorem 3.21 here. However, we cannot because $B \triangleright \overline{A}$ will not have MWC-density relative to A. To fix this, we will mimic the proof of Theorem 3.21; that is, we shall construct a B-computable monotone selection function g_B such that $g_B(\overline{A}) = f(A \sqcup (B \triangleright \overline{A}))$. Then $\overline{A} \triangleleft g_B(\overline{A})$ will be a superset of $(B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))$ with density $1 - \alpha$ because A has MWC-density α relative to B. Then there is some X such that

$$X \triangleright (\overline{A} \triangleleft g_B(\overline{A})) = (B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A})).$$

Finally, it suffices to construct an A-computable monotone selection function g_A such that $B \triangleleft g_A(B) = X$. The set X will then have density β due to the fact that B has MWC-density β relative to A, and Lemma 3.7 will ensure that

$$\rho((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \rho(X \triangleright (\overline{A} \triangleleft g_B(\overline{A}))) = \beta(1 - \alpha),$$

as desired.

Define $g_B : 2^{<\omega} \to \{0,1\}$ via $g_B(\sigma) = 1$ if and only if $f_B(\overline{\sigma}) = 1$, where $\overline{\sigma}$ is defined to be $\tau \in 2^{|\sigma|}$ with $\tau(k) = 1$ if and only if $\sigma(k) = 0$. Then

$$g_B(\overline{A}) = f_B(A) = f(A \sqcup (B \triangleright \overline{A})).$$

Let $g_A : 2^{<\omega} \to \{0,1\}$ be defined via $g_A(\sigma) = f(A \sqcup (\sigma \triangleright \overline{A})$, where $A \sqcup (\sigma \triangleright \overline{A}) = \tau \in 2^{p_{\overline{A}}(|\sigma|)}$ is defined via $\tau(n) = 1$ if and only if $n \in A$ or $n = p_{\overline{A}}(k)$ for some $k < |\sigma|$ and $\sigma(k) = 1$. We now claim that $B \triangleleft g_A(B) = X$.

Recall that X is

$$((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) \triangleleft (\overline{A} \triangleleft g_B(\overline{A})).$$

As mentioned above, $g_B(\overline{A}) = f(A \sqcup (B \triangleright \overline{A}))$, so we may apply Lemma 3.19 to obtain

$$X = (B \triangleright \overline{A}) \triangleleft (\overline{A} \cap f(A \sqcup (B \triangleright \overline{A}))).$$

Suppose $n \in X$. By the definition of X,

$$p_{\overline{A} \cap f(A \sqcup (B \triangleright \overline{A}))}(n) \in B \triangleright \overline{A}.$$

That is, the n-th element of $\overline{A} \cap f(A \sqcup (B \triangleright \overline{A}))$ is in $B \triangleright \overline{A}$. Therefore, it is of the form $p_{\overline{A}}(b_k)$ for some k. Furthermore, $p_{\overline{A}}(b_k) \in f(A \sqcup (B \triangleright \overline{A}))$, so by definition, $f((A \sqcup (B \triangleright \overline{A})) \upharpoonright p_{\overline{A}}(b_k)) = 1$. This then implies, by the definition of g_A, that $g_A(B \upharpoonright b_k) = 1$. Therefore, $b_k \in g_A(B)$, and $p^{-1}_{g_A(B)}(b_k) \in B \triangleleft f_A(B)$. Finally, note that $p^{-1}_{g_A(B)}(b_k) = n$, because every element of $g_A(B)$ is an element of $\overline{A} \cap f(A \sqcup (B \triangleright \overline{A}))$ by definition, and b_k corresponds to the n-th such one. Therefore, $n \in B \triangleleft g_A(B)$. This argument reverses, so $B \triangleleft g_A(B) = X$. □

Note that if A has MWC-density α relative to B and $B \triangleleft \overline{A}$ and B has MWC-density β relative to A, then Theorem 3.27 can be obtained as an easy corollary of Lemma 3.28. We have

$$A \cup B = A \sqcup (B \cap \overline{A}) = A \sqcup ((B \triangleleft \overline{A}) \triangleright \overline{A}).$$

Whether this latter relativization is implied by the other conditions is essentially Question 4.9.

We are ready to attempt to lift Theorem 3.13 to MWC-density. Proposition 3.15 relativizes in straightforward fashion. As a result, the proof of Lemma 3.12 immediately lifts to prove an analog for MWC-density. There is a disjoint sequence of sets $\{A_i\}_{i\in\omega}$ such that each A_i has MWC-density $\frac{1}{2^{i+1}}$ relative to the others. (Theorem 3.21 requires more relativization than Theorem 3.9, but the fact that Theorem 3.21 itself relativizes ensures that the same proof technique applies.)

Unfortunately, the fact that unions do not preserve MWC-density in general means that given a real r, we do not know that the infinite union of the A_i's corresponding to the binary expansion of r will have MWC-Density. In the finite case, however, Lemma 3.28 will ensure the union has the desired MWC-density.

Proposition 3.29. Let X be 1-Random and let $\{A_i\}_{i\in\omega}$ be constructed from X as in Lemma 3.12. If D is a finite set of natural numbers, then $\bigsqcup_{i\in D} A_i$ is X-computable and has MWC-density $\Sigma_{i\in D}\frac{1}{2^{i+1}}$.

Proof. Essentially, each $\bigsqcup_{i\in D} A_i$ is composed of finitely many unions of the form found in Lemma 3.28 and finitely many applications of the `into` operation. Van Lambalgen's theorem will ensure that we have all of the necessary relativizations so that we can use Lemma 3.28 and Theorem 3.21 to reduce the number of unions by one. Combined with induction on the size of the union, this will prove the result.

Recall that we defined $A_0 = \overline{X^{[0]}}$ and

$$A_i = \overline{X^{[i]}} \rhd X^{[i-1]} \rhd \ldots \rhd X^{[0]}$$

for $i > 0$. Therefore,

$$\bigsqcup_{i\in D} A_i = \bigsqcup_{i\in D} \overline{X^{[i]}} \rhd X^{[i-1]} \rhd \ldots \rhd X^{[0]}.$$

(If $i = 0$ or $i = 1$ then we take $X^{[i-1]} \rhd \ldots \rhd X^{[0]}$ to mean ω and $X^{[0]}$ respectively to ensure that this does indeed match the definition of A_i from Lemma 3.12.)

We argue by induction on the size of D. If D is a singleton, then its member is of the form $\overline{X^{[i]}} \rhd X^{[i-1]} \rhd \ldots \rhd X^{[0]}$ for some i. By Van Lambalgen's Theorem, each $X^{[j]}$ is 1-Random relative to the join of the other. Therefore, by Proposition 3.15 each has MWC-density $\frac{1}{2}$ relative to the

join of the others. Thus, $\overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}$ has MWC-density $\frac{1}{2^{i+1}}$, by Theorem 3.21. This concludes the base case.

Now, suppose it holds that for any 1-Random X and any finite set D of size less than or equal to n, $\bigsqcup_{i \in D} A_i$ has MWC-density $\Sigma_{i \in D} \frac{1}{2^{i+1}}$. Let D have size $n+1$. First, consider the case when $0 \in D$. Then using the fact that $(A \sqcup B) \triangleright C = (A \triangleright C) \sqcup (B \triangleright C)$ and the associativity of the `into` operation,

$$\bigsqcup_{i \in D} A_i = \overline{X^{[0]}} \sqcup (\bigsqcup_{i \in D, i > 0} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}) =$$

$$\overline{X^{[0]}} \sqcup ((\bigsqcup_{i \in D, i > 0} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots X^{[1]}) \triangleright X^{[0]})$$

Let Y be defined via $Y^{[i]} = X^{[i+1]}$. Y is 1-Random relative to $X^{[0]}$, by Van Lambalgen's Theorem. Thus, by the relativized induction hypothesis,

$$\bigsqcup_{i \in D, i > 0} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots X^{[1]} = \bigsqcup_{i \in D, i > 0} \overline{Y^{[i-1]}} \triangleright Y^{[i-2]} \triangleright \ldots Y^{[0]}$$

has MWC-density $\Sigma_{i \in D, i > 0} \frac{1}{2^i}$ relative to $X^{[0]}$. Finally, Lemma 3.28 then implies that

$$\overline{X^{[0]}} \sqcup ((\bigsqcup_{i \in D, i > 0} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots X^{[1]}) \triangleright X^{[0]})$$

has MWC-density

$$\frac{1}{2} + (\sum_{i \in D, i > 0} \frac{1}{2^i})(1 - \frac{1}{2}) = \frac{1}{2} + (\sum_{i \in D, i > 0} \frac{1}{2^{i+1}}) = \sum_{i \in D} \frac{1}{2^{i+1}}.$$

Now, suppose that $j > 0$ is the least element of D. Then we have

$$\bigsqcup_{i \in D} A_i = (\overline{X^{[j]}} \triangleright X^{[j-1]} \triangleright \ldots \triangleright X^{[0]}) \sqcup (\bigsqcup_{i \in D, i > j} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}) =$$

$$(\overline{X^{[j]}} \triangleright (X^{[j-1]} \triangleright \ldots \triangleright X^{[0]})) \sqcup (\bigsqcup_{i \in D, i > j} (\overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[j]}) \triangleright (X^{[j-1]} \triangleright \ldots \triangleright X^{[0]})) =$$

$$(\overline{X^{[j]}} \sqcup (\bigsqcup_{i \in D, i > j} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[j]})) \triangleright (X^{[j-1]} \triangleright \ldots \triangleright X^{[0]}).$$

Let Y be defined via $Y^{[i]} = Y^{[i+j]}$ and $\hat{D} = \{n - j : n \in D\}$. Then Y is 1-Random by Van Lambalgen's Theorem and $\hat{D}$ is a set of size n that contains 0. Therefore, we can apply the

relativized version of the previous case to see that

$$\overline{X^{[j]}} \sqcup (\bigsqcup_{i\in D, i>j} \overline{X^{[i]}} \rhd X^{[i-1]} \rhd \ldots \rhd X^{[j]}) = \overline{Y^{[0]}} \sqcup (\bigsqcup_{i\in \hat{D}, i>0} \overline{Y^{[i]}} \rhd Y^{[i-1]} \rhd \ldots \rhd Y^{[0]})$$

has MWC-density

$$\sum_{i\in \hat{D}} \frac{1}{2^{i+1}} = \sum_{i\in D} \frac{1}{2^{i+1-j}} = \sum_{i\in D} \frac{2^j}{2^{i+1}}$$

relative to $X^{[j-1]} \rhd \ldots \rhd X^{[0]}$. Since $X^{[j-1]} \rhd \ldots \rhd X^{[0]}$ has MWC-density $\frac{1}{2^j}$ relative to $\overline{X^{[j]}} \sqcup (\bigsqcup_{i\in D, i>j} \overline{X^{[i]}} \rhd X^{[i-1]} \rhd \ldots \rhd X^{[j]})$, by Van Lambalgen's Theorem and multiple iterations of the relativized form of Theorem 3.21, it follows that

$$(\overline{X^{[j]}} \sqcup (\bigsqcup_{i\in D, i>j} \overline{X^{[i]}} \rhd X^{[i-1]} \rhd \ldots \rhd X^{[j]})) \rhd (X^{[j-1]} \rhd \ldots \rhd X^{[0]})$$

has MWC-density

$$(\sum_{i\in D} \frac{2^j}{2^{i+1}}) \frac{1}{2^j} = \sum_{i\in D} \frac{1}{2^{i+1}}.$$

This completes the induction. □

Thus while our current technology does not give us the analog of Theorem 3.13 for every $r \in (0, 1)$, we do obtain it for those r that are finite sums of powers of two.

Unfortunately, it remains open whether or not this can be extended to infinite unions of this form, which is Question 4.10 below. The difficulty lies once again in the fact that the input set can change which bits are and are not selected. In theory, given any $0 < r < 1$ and the set coding its binary expansion B_r as in Theorem 3.13, for any $\epsilon > 0$, there exists an N such that $\bigsqcup_{n\in B_r, n<N} A_n \lhd f(\bigsqcup_{n\in B_r, n<N} A_n)$ has MWC-density within ϵ of r. If we could impose a nice enough uniformity condition on the A_n's, then we might be able to assert that the change from adding the remaining A_n's is no more than ϵ. In practice, however, elements of A_k may change which bits are selected by f in non-uniform fashion so that the density of $\bigsqcup_{n\in B_r, n<N} A_n \lhd f(\bigsqcup_{n\in B_r, n<N} A_n)$ is meaningless compared to the density of $\bigsqcup_{n\in B_r} A_n \lhd f(\bigsqcup_{n\in B_r} A_n)$.

CHAPTER 4

CLOSING REMARKS

4.1 Review and Future Work

In the first half of this dissertation, we studied the intrinsically small sets in detail. We proved some useful closure properties for them, and proved that hyperimmunity and intrinsic smallness are not the same anywhere in the Turing degrees. We partially filled out the diagram of relationships between various notions of intrinsic computation. However, there are still many open questions. Answering all potential remaining implications and nonimplications is a clear candidate for future work. We also improved upon the known results about relativized intrinsic density, including a complete description of the X-intrinsically small sets for all X.

In the second half, we set out to separate intrinsic density and randomness. To do this, we defined the `into` and `within` operations. These turned out to be useful tools for coding sets in noncomputable fashion. They formed a calculus of sorts for intrinsic density, which allowed us to construct sets of arbitrary intrinsic density from any 1-Random. For almost all r, this constructed set could not compute a set random with respect to μ_r, showing that intrinsic density is much weaker computationally. We then compared and contrasted intrinsic density to the more well-known notion of MWC-density. While `into` and `within` illustrated the similarities between the notions, the join and the union highlighted their differences.

There is significant room for future work. We did not investigate full KL-density in this paper, nor did we investigate how it compares to intrinsic density and MWC-density. Additionally, it is unknown how `into` behaves with randomness: if A is μ_r-random relative to B and B is μ_s-random relative to A, is $B \triangleright A$ μ_{rs}-random? A simple probabilistic argument supports this, but a formal proof is not immediately obvious. So far, we have exploited the fact that stochasticity is determined by analyzing the asymptotic density of sets defined using the `within` operation and utilizing the connections between `into`, `within`, and Lemma 3.7. Different methods seem to be necessary to

study how the `into` operation interacts with martingales and/or ML-tests.

We conclude by compiling some specific open questions.

4.2 Open Questions

Recall the following unresolved question from Section 2.1.

Question 4.1. If A is intrinsically small and f is a total computable injective function whose range has defined density, then is $f(A)$ intrinsically small?

Additionally, the natural follow-up question to Corollary 2.11 remains open. This question is closely related to Question 4.1.

Question 4.2. Suppose that A is an intrinsically small set. Is A small for the class of total computable *-injective functions? Total computable injective functions?

Notice that if the answer here is yes, then the analogue of Corollary 2.6 for computable injective functions follows immediately from the same argument. Therefore, a positive answer yields a positive answer to Question 4.1, and a negative answer to Question 4.1 yields a negative answer to Question 4.2. The opposite direction also seems closely related, but no implications are immediately obvious. [1]

We say that X is *range stochastic* for r if $\rho(f(A) \triangleleft range(f)) = r$ for all total computable injective functions f.

Question 4.3. Is it the case that every set of intrinsic density α has range-density α? That is, for any set A with intrinsic density α, is it the case that $f(A) \triangleleft range(f)$ has density α for all total computable injective functions f?

Note that this is similar to Question 4.1.

Recall that we proved there is an intrinsically coarsely computable set that is not intrincially generically computable. The reverse separation is still open.

Question 4.4. Is there an intrinsically generically computable set which is not intrinsically coarsely computable?

[1] Theorems 2.9 and 2.10 help to characterize what must happen in the scenario where the answer to Question 4.2 is no: The upper and lower density of the range are relatively far apart, allowing small elements of $f(A)$ to show up at late stages after any computable process "thinks" range(f) is done enumerating small elements.

One potentially useful result to apply to this question is the result of Arslanov [2] that the only c.e. DNC degree is $\emptyset'$. We know from [3] that the degrees that contain an intrinsically small set are those which are high or DNC. Since the domain of an intrinsic generic description is c.e. and can compute an intrinsically small set (its complement), its degree must be high or DNC. Therefore, it is high.

Recall that we have a test for intrinsic smallness using principal functions, but not a characterization.

Question 4.5. Is it the case that if A is intrinsically small, then p_A is not weakly computably traced by $h = \lambda n(n!)$? If it is not the case, is there an intrinsically small set which does not dominate h? (This would mean that $p_A(n) \leq n!$ infinitely often.)

Of course there are computably dominated intrinsically small sets, by Theorem 2.3. However, it is not clear if there are any "nice" computable functions (i.e. something naturally occurring in arithmetic or combinatorics) that dominate an intrinsically small set. It is not even clear if there are such functions that are not dominated by the principal function of an intrinsically small set. Our usual strategy for constructing intrinsically small sets is no help for this, as it requires arbitrarily large witnesses.

Above, we proved that the degrees of intrinsic density r sets are closed upwards. However, it remains to be seen exactly what those degrees are.

Question 4.6. Which Turing degrees contain a set of intrinsic density r?

We know that this cannot be the high or DNC degrees in general as is the case for the intrinsically small sets: as pointed out by Denis Hirschfeldt, there are uncountably many reals and only countably many sets in each Turing degree, so no degree can have a representative for each r. Furthermore, it cannot be a subset of the r-high or r-DNC degrees in general because for almost all r we constructed an intrinsic density r set which is r-computable.

For intrinsic density, we proved that $P(A \triangleleft C) = \alpha$ if C is computable and $P(A) = \alpha$. It is known that $C \triangleleft A$ does not necessarily have intrinsic density in general, as witnessed by $A = \omega$. This leads to the following question.

Question 4.7. Are there conditions on A guaranteeing that $C \triangleleft A$ has intrinsic density for computable C?

For a discussion of the applications of this question, see the remark following the proof of Proposition 3.11.

In proving Theorem 3.21, we used more relativization than was necessary in the intrinsic density analog Theorem 3.9. However, it is not known whether this is necessary or merely useful.

> **Question 4.8.** Is the relativization optimal in Theorem 3.21? That is, are there sets A and B such that B has MWC-density β relative to A and A has MWC-density α, but $B \triangleright A$ does not have MWC-density $\alpha\beta$?

The same proof that showed the relativization used in Theorem 3.9 is optimal will not work for Theorem 3.21 because $A \oplus \overline{A}$ will not have MWC-density, by Proposition 3.23.

We could not directly lift the proof that the intersection of two intrinsically dense sets multiplied the intrinsic densities of the sets to the case of MWC-density due to different relativization requirements between Theorem 3.9 and its analog Theorem 3.21. A positive resolution to the following question would allow us to do this.

> **Question 4.9.** If A has MWC-density α and C is computable, does A have MWC-density α relative to $C \triangleleft A$? If so, does this relativize? If this is not true, is it at least the case that whenever A has MWC-density α relative to B and B has MWC-density relative to A, then A has MWC-density α relative to $B \triangleleft A$?

Our usual techniques do not suffice to answer this question, since they use oracles, or relativized information, to answer questions about non-relativized MWC-density. This question requires us to answer a question about MWC-density relative to a specific set using non-relativized information.

If this is true and relativizes, or the weaker formulation is true, then whenever A has MWC-density α relative to B and B has MWC-density β relative to A, Theorem 3.21 would imply $A \cap B$ has MWC-density $\alpha\beta$ and Lemma 3.28 would imply $A \cup B$has MWC-density $\alpha + \beta - \alpha\beta$. (Recall that both of these facts are true, but they required separate proofs.)

At the end of Section 3.3, we discussed the difficulty in translating the proof of Corollary 3.13 into the intrinsic density case.

> **Question 4.10.** Given a sequence $\{A_n\}_{n\in\omega}$ as constructed in Lemma 3.12, let $0 < r < 1$ and let B_r be the set representing its binary sequence. Does $\bigsqcup_{n\in B_r} A_n$ have MWC-density r? If this is not so in general, are there additional requirements we can put on the sequence to guarantee this?

APPENDIX A

NOTATION

A.1 Strings and Sets

We shall use ω to represent the natural numbers, which include 0. We use ω^ω to denote the set of all infinite strings of natural numbers, and $\omega^{<\omega}$ is the set of all finite strings of natural numbers. For natural numbers n, n^ω and $n^{<\omega}$ will denote the set of $\{0, \ldots, n-1\}$-valued infinite strings and finite strings, respectively. Of particular importance are 2^ω, the set of infinite binary strings, and $2^{<\omega}$, the set of finite binary strings.

The following notational conventions all apply unless otherwise stated. We shall use lowercase English letters such as b to represent natural numbers, and uppercase English letters such as B shall represent infinite sets of natural numbers. We shall write $\overline{B}$ to represent the complement of B within the natural numbers. Given a set B, we shall use subscripted lowercase letters to represent its elements in increasing order, i.e. $B = \{b_0 < b_1 < b_2 < \cdots < b_n < \ldots\}$. The *principal function* of B, denoted by $p_B : \omega \to B$, is defined via $p_B(n) = b_n$. The *characteristic function* of B, denoted by $\chi_B : \omega \to \{0, 1\}$, is defined via $\chi_B(n) = 1$ if and only if $n \in B$. We will often abuse notation and write $B(n)$ to mean $\chi_B(n)$. For finite sets, there is a canonical coding that associates a finite set D_n with a natural number n, for which the k-th bit of the binary expansion of n is 1 if and only if $k \in D_n$. As an example, the binary expansion of 11 is 1011. Therefore, $D_{11} = \{0, 1, 3\}$.

Lowercase Greek letters such as σ will be used to represent finite strings from $\omega^{<\omega}$ or $n^{<\omega}$. We shall use $|\sigma|$ to represent the length of σ. We think of sets as elements of 2^ω by identifying them with their characteristic function, and we think of functions from ω to ω as elements of ω^ω in the obvious way. For $f \in \omega^\omega$ and a natural number n, $f \upharpoonright n$ is the restriction of f to the domain $\{0, 1, \ldots, n-1\}$, which is an element of $\omega^{<\omega}$. Given $\sigma \in \omega^{<\omega}$ and $f \in \omega^\omega$, we say $\sigma \preceq f$ if $\sigma = f \upharpoonright |\sigma|$. This applies to sets via the identification with characteristic functions. We will occasionally use lambda notation to describe functions, where $f = \lambda n(f(n))$ is the same as saying f is the function from ω to ω that

takes n to $f(n)$, where $f(n)$ is some fixed expression.

Baire space is the topological space obtained by equipping ω^ω with the topology generated by the basic open sets $[\sigma] = \{f \in \omega^\omega : \sigma \preceq f\}$. Cantor space is obtained by defining the analogous topology on 2^ω.

A.2 Turing Computation

We shall only list the notation we shall use for computation here. Familiarity with the concept of Turing computation and its properties is assumed. For a review of these concepts, one may consult Soare [24].

We use φ_e to represent the Turing machine associated with the code e under some fixed enumeration of all Turing machines. Our formal convention will be that our Turing machines take as input a single natural number and output a single natural number, however informally we may use some pairing function $\langle , \rangle$ or coding system to replace natural numbers with other nice countable objects such as finite strings of natural numbers or finite binary strings. We use $\varphi_{e,s}(x)$ to represent running the e-th Turing machine s steps with input x. Here we use $\varphi_{e,s}(x) \downarrow = y$ to mean that the e-th Turing machine halts on input x after no more than s steps and outputs y. Otherwise, we write $\varphi_{e,s}(x) \uparrow$. We say $\varphi_e(x)$ *converges* to y if there is some s with $\varphi_{e,s} \downarrow = y$, and we write $\varphi_e(x) \downarrow$ to mean there is some y to which $\varphi_e(x)$ converges. If there is no such y, we say $\varphi_e(x)$ *diverges* and write $\varphi_e(x) \uparrow$.

We let $W_e = \{n \in \omega : \varphi_e(n) \downarrow\}$. A set X is said to be *computably enumerable*, or c.e., if $X = W_e$ for some e.

We use Φ_e^σ to represent the e-th oracle Turing machine with a finite binary string σ fed in on the oracle tape. The above notation for convergence and divergence is used for Φ_e^σ in the obvious way, with the caveat that $\Phi_e^\sigma(x) \downarrow$ only if the the machine does not query the oracle tape beyond the length of σ. As shifting the head of the oracle tape counts as a step in our computation, $\Phi_e^\sigma(x) \downarrow$ implies $\Phi_{e,|\sigma|}^\sigma(x) \downarrow$. Given a set X, we say that $\Phi_e^X(x) \downarrow = y$ if and only if there exists a $\sigma \preceq X$ with $\Phi_e^\sigma(x) \downarrow = y$.

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