Separating Stochasticity and Randomness

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Context

We work in Cantor space, 2^ω. Unless otherwise stated, we assume all sets are infinite.

We are interested in sets which are random with respect to the r-Bernoulli measures μ_r for real numbers r: given a finite binary string σ, let $|\sigma|$ denote the length of σ, and let $\#\sigma$ denote the number of 1's in σ. Then

$$\mu_r([\sigma]) = r^{\#\sigma}(1-r)^{|\sigma|-\#\sigma}$$

$\mu_{\frac{1}{2}}$ is the Lebesgue measure on the unit interval. When we say μ_r-random, we mean μ_r-ML-random, i.e. not contained in any u.c.e. null set relative to μ_r.

Asymptotic Density

Definition

For a set $X \in 2^\omega$ and $n \in \omega$,

$$\rho_n(X) = \frac{\#(X \upharpoonright n)}{n}$$

Then

$$\overline{\rho}(X) = \limsup_{n\to\infty} \rho_n(X), \quad \underline{\rho}(X) = \liminf_{n\to\infty} \rho_n(X)$$

are called the upper density of X and the lower density of X respectively. If $\overline{\rho(X)} = \underline{\rho(X)}$, then we call this value the (asymptotic) density of X and denote it by $\rho(X)$.

The **law of large numbers** says that $\rho(X) = r$ for any set X which is μ_r-random.

Sampling

Asymptotic density, or the law of large numbers, is not a particularly random property: the set of even numbers has density $\frac{1}{2}$, but is clearly not random. However, sufficiently random sets satisfy a stronger property: the density remains the same even if we sample subsets in some predictable fashion. Conversely, if we can sample in order to obtain a sequence of different density, the original sequence was not random.

We think of this as having flipped infinitely many r-biased coins and hiding them underneath cups. We may then re-arrange or remove coins in an attempt to, after revealing the coins, obtain a sequence of different asymptotic density.

Stochasticity

A set X is r-Church stochastic if, whenever we are allowed to remove cups following some total computable process, where we are allowed to look at the first n coins to determine whether or not we should remove the $n + 1$-st coin, the resulting density is r whenever the sequence is infinite.

A set X is r-MWC stochastic if the density is r for every infinite sequence selected by computable processes like those for Church stochasticity, however the processes need not be total.

X is r-KL stochastic if the density is r for partial computable processes which are allowed to check and remove coins out of order, but are still required to base their next decision only on the values of previously checked coins.

Intrinsic Density

A special case of KL-stochasticity is injection stochasticity, which chooses the same subsequence non-monotonically regardless of coin value. (In other words, it is uniform KL-stochasticity.)

Astor proved that r-injection stochasticity corresponds with intrinsic density r.

Definition

The absolute upper density of X is

$$\overline{P}(X) = \sup_{\pi}\{\overline{\rho}(\pi(X)) : \pi \text{ a computable permutation}\}$$

The absolute lower density, $\underline{P}(X)$, is defined similarly for the lim inf. If these are equal, we denote the quantity by $P(X)$ and call it the intrinsic density of X. We use P_Y to denote intrinsic density relative to Y.

Intrinsic Density and Randomness

If X is μ_r-random, then the fact that X satisfies the law of large numbers and the fact that the class of μ_r-randoms is closed under computable permutation proves that $P(X) = r$.

However, it is not hard to see that the converse is false. If A has intrinsic density $\frac{1}{2}$, then $A \oplus A$ has intrinsic density $\frac{1}{2}$ as well. However, $A \oplus A$ is not random by Van Lambalgen's Theorem, which says that $A \oplus B$ is only random if A and B are relatively random to each other.

Separating Randomness and Stochasticity

While this shows that there are sets which have intrinsic density but are not random, it is a structural fact. If A is random, $A \oplus A$ can still trivially compute A, and therefore can compute a random. We have not shown any difference in computational properties between the two notions.

This will be our goal. We shall prove that, for almost all r, there is an r-computable set of intrinsic density r. As a corollary, this set will not be able to compute any μ_r-random set, as a set random with respect to X cannot be X computable. To do this, we first need to develop some machinery.

The Into Operation

Definition

Given two sets

$$A = \{a_0 < a_1 < a_2 < \dots\}$$

and

$$B = \{b_0 < b_1 < b_2 < \dots\}$$

we define the set $B \triangleright A$, or "B into A," to be

$$\{a_{b_0} < a_{b_1} < a_{b_2} < \dots\}$$

As an example: If E is the set of even numbers and T is the set of multiples of three, then $E \triangleright T$ is the set of multiples of six.

Connection Outside Computability

Y is said to preserve normality if, for all X, X is 2-normal implies $X \triangleright Y$ is 2-normal.

Theorem (Kamae-Weiss, 1973)

If $\lim_{n\to\infty} \frac{y_n}{n} < \infty$, then Y preserves normality if and only if Y is completely deterministic. (This means a specific form of entropy is zero.)

Lemma

$\rho(B \triangleright A) = \rho(B)\rho(A)$

Proof Sketch.

We will be able to estimate the limsup for $B \triangleright A$ with

$$\limsup_{n\to\infty} \frac{n}{a_n+1} \leq \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} \cdot \frac{b_n+1}{b_n+1} \leq (\limsup_{n\to\infty} \frac{b_n+1}{a_{b_n}+1})(\limsup_{n\to\infty} \frac{n+1}{b_n+1})$$

The latter term will be the limsup for B, and the former will be no greater than the limsup for A. The liminf will be similar, and as both B and A have density we are done. $\square$

`Into` and Intrinsic Density

The following theorem is central to our argument. We shall use it to combine sets and manipulate intrinsic density.

Theorem
If A has intrinsic density α and B has intrinsic density β relative to A, then $B \triangleright A$ has intrinsic density $\alpha\beta$.

To prove this, we need to introduce another operation.

The `Within` Operation

Definition

Given two sets
$$A = \{a_0 < a_1 < a_2 < \dots\}$$
and
$$B = \{b_0 < b_1 < b_2 < \dots\}$$
we define the set $B \lhd A$, or "B within A," to be
$$\{n : a_n \in B\}$$

In other words, $B \cap A \subseteq A$, so there is some set X such that $X \rhd A = A \cap B$. We use $B \lhd A$ to represent this set X. As an example, if S is the set of multiples of 6, and T is the set of multiples of 3, $S \lhd T = E$, the set of even numbers.

Basic Properties

- $A = A \rhd \omega$
- $A = \omega \rhd A$
- $(B \rhd A) \sqcup (\overline{B} \rhd A) = A$
- $\rhd$ is associative
- $\omega = A \lhd A$

- $(B \lhd A) \sqcup (\overline{B} \lhd A) = \omega$
- If $B \subseteq A$, $(B \lhd A) \rhd A = B$.
- $\lhd$ is not associative
- $(B \rhd A) \lhd A = B$
- $A \oplus B = (A \rhd E) \sqcup (B \rhd O)$

A Set Calculus

In proving theorems using these operations, we often use technical lemmas about their behavior which say nothing about stochasticity or randomness, nor even computability.

Lemma

For any sets A, B, C:

- $\rho(B \triangleright A) = \rho(B)\rho(A)$.
- $(A \triangleleft C) \triangleleft (B \triangleleft C) = A \triangleleft (B \cap C)$.
- $(A \triangleleft C) \triangleleft (B \triangleleft C) = (A \triangleleft B) \triangleleft (C \triangleleft B)$.
- *If $C \cap E$ is infinite and coinfinite and*

$$\rho((A \triangleright E) \triangleleft (C \cap E)) = \rho((B \triangleright \overline{E}) \triangleleft (C \cap \overline{E})) = r$$

then $\rho((A \oplus B) \triangleleft C) = r$

Within and Intrinsic Density

Theorem

If C is computable and $P(A) = \alpha$, $P(A \triangleleft C) = \alpha$.

Proof Sketch.

Suppose not. Then, without loss of generality, there is a computable permutation π such that $\rho(\pi(A \triangleleft C)) > \alpha$. In addition, there is the map $f_C : C \to \omega$ such that $f_C(c_n) = n$, and $\pi(f(A)) = \pi(A \triangleleft C)$. However, f_C is not a permutation of ω.

Using f_C, construct a computable permutation π_C such that π_C agrees with π everywhere but a small computable subset of C, which is used to turn f_C into a permutation. Then π_C will witness $\rho(\pi_C(A)) = \rho(\pi(A \triangleleft C)) > \alpha$, so $P(A) \neq \alpha$. $\square$

Back to the `Into` Theorem

Theorem

If A has intrinsic density α and B has intrinsic density β relative to A, then $B \triangleright A$ has intrinsic density $\alpha\beta$.

Proof Sketch.

Assume $P(A) = \alpha$, and suppose the theorem fails, i.e. without loss of generality there is a computable permutation π such that $\rho(\pi(B \triangleright A)) > \alpha\beta$.

Apply the same technique as in the previous theorem, using the fact that B has intrinsic density relative to A to construct an A-computable permutation π_A which witnesses that $\rho(\pi_A(B)) = \rho(\pi(B \triangleright A) \triangleleft \pi(A)) > \beta$. Therefore $P_A(B) > \beta$. $\square$

Intersections

Corollary

If A has intrinsic density α and B has intrinsic density β relative to A, then $A \cap B$ has intrinsic density $\alpha\beta$.

Proof.

By the relativized form of the `Within` theorem, $B \triangleleft A$ has intrinsic density β relative to A. By the `Into` theorem, $A \cap B = (B \triangleleft A) \triangleright A$ has intrinsic density $\alpha\beta$. $\square$

Unions

We shall construct sets using the `Into` theorem, then combine them using disjoint unions.

Theorem (Essentially Jockusch and Schupp, 2012)

Suppose $\{S_i\}_{i\in\mathbb{N}}$ *is a countable sequence of sets. If all of the following occur:*

- *The* S_i*'s are disjoint*
- S_i *has positive intrinsic density for all* i
- *the limit of the density of the tail of this sequence goes to* 0 *as* i *goes to infinity*

then the union has defined intrinsic density and it is the sum of the densities of the S_i*'s.*

Powers of Two

Theorem (Van Lambalgen, 1990)

Any set random with respect to $\mu_{\frac{1}{2}}$ can be decomposed into countably many sets which are random with respect to $\mu_{\frac{1}{2}}$ relative to any combination of the others.

Theorem

There is a countable, disjoint sequence of sets $\{A_i\}_{i\in\omega}$ such that A_i has intrinsic density $\frac{1}{2^{i+1}}$. Furthermore, this satisfies the requirements of Jockusch and Schupp's result.

Proof Sketch.

Let X be 1-Random. Then by Van Lambalgen's theorem and the fact that 1-Randoms have intrinsic density $\frac{1}{2}$, we have countably many sets X_n all with intrinsic density relative to any combination of the rest. Then define B_0 to be all of the naturals, $A_n = \overline{X_n} \triangleright B_n$, and $B_{n+1} = X_n \triangleright B_n$. $\square$

Avoiding Randomness

Theorem

If r is random with respect to $\mu_{\frac{1}{2}}$, then r computes a set of intrinsic density r.

Proof.

Let r be random with respect to $\mu_{\frac{1}{2}}$ and let B_r be the set corresponding to its binary expansion to avoid confusion. We use B_r in place of X to construct the sequence from the previous theorem.

Then by the theorem of Jockusch and Schupp, $A = \bigcup_{n \in B_r} A_n$ will have intrinsic density the sum of the densities of the A_n's for $n \in B_r$. However, as B_r is the binary expansion of r and each A_n has intrinsic density $\frac{1}{2^{n+1}}$, these sum to r, completing the proof. $\square$

Note that no μ_r-random set can be r-computable, so this set cannot compute any μ_r-random set.

MWC and Church Stochasticity

What about other notions of stochasticity? It is natural to ask if the above techniques can be used to provide a similar separation between randomness and other notions of stochasticity.

However, there are some key structural differences between MWC and Church stochasticity and injection stochasticity which will cause the above argument to fail.

We will state our results in terms of MWC stochasticity, but they will apply to Church stochasticity as well.

Differences in Stochasticity

Lemma

If A is MWC stochastic for some real other than 1, then $A \oplus A$ is not MWC stochastic.

Lemma

There exist disjoint sets A and B such that both are 0-MWC stochastic, but $A \sqcup B$ is not MWC stochastic.

Lemma

(Bienvenu) There exist disjoint sets A and B such that both are $\frac{1}{2}$-MWC stochastic, but $A \sqcup B$ is not MWC stochastic.

Similarities in Stochasticity

Theorem

If A is r-MWC stochastic and C is computable, then $A \triangleleft C$ is r-MWC stochastic.

Theorem

If A is α-MWC stochastic relative to B and B is β-MWC stochastic relative to A, then $B \triangleright A$ is $\alpha\beta$-MWC stochastic.

This second theorem looks similar to the `Into` theorem for intrinsic density. However, notice that there is an extra relativization requirement on A. It is unknown if this is necessary.

Theorem

If A is α-MWC stochastic relative to B and B is β-MWC stochastic relative to A, then $A \cap B$ is $\alpha\beta$-MWC stochastic.

Unions

Theorem

If A is α-MWC stochastic relative to B, and B is β-MWC stochastic relative to A, then $A \cup B$ is $\alpha + \beta - \alpha\beta$-MWC stochastic.

Theorem

If A is α-MWC stochastic relative to B, and B is β-MWC stochastic relative to A, then $A \sqcup (B \triangleright \overline{A})$ is $\alpha + \beta - \alpha\beta$-MWC stochastic.

Constructing MWC stochastic sets

Theorem

Every $\mu_{\frac{1}{2}}$-random set X computes a set of r-MWC stochasticity, where r is any real in the unit interval equal to a finite sum of powers of 2.

Proof.

We construct A_n and B_n from X as in the intrinsic density case. Van Lambalgen's Theorem ensures we have the relativization necessary even for the additional requirements. Then we verify that any finite union of the A_n's is of the form necessary to apply the second union theorem. $\square$

Some Open Questions

Question

For r nonrandom, is there an intrinsic density r set which cannot compute a μ_r-random set?

Question

Given r, is there an r-MWC-stochastic set which cannot compute a μ_r-random set?