Origami and Mathematics

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What are "nice" properties of paper?

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- Cannot be sheared.
- It can be easily folded!
Something else...
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You can trisect an angle by folding paper!
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This shows how origami is more powerful than straightedge and compass.
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\[ \alpha \rightarrow \beta \]
You can trisect an angle by folding paper!

This shows how origami is more powerful than straightedge and compass.
How to do it?
How to do it?
What is happening?

We are finding simultaneous tangents to parabolas.
An algebra application: solving $x^3 + ax + b$

We will find the solutions for $x^3 + ax + b = 0$ where $a, b \in \mathbb{R}$ and $b \neq 0$ by finding a simultaneous tangent to:

$\left(y - \frac{1}{2}a\right)^2 = 2bx$  \hspace{1cm} \text{and} \hspace{1cm}  y = \frac{1}{2}x^2$
An algebra application: solving $x^3 + ax + b$

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The slopes are:

$$m_1 = \frac{b}{y_1 - \frac{1}{2}a} \quad \text{and} \quad m_2 = x_2$$
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Hence:

$$x_1 = \frac{b}{2m_1^2} \quad \text{and} \quad x_2 = m_2$$

$$y_1 = \frac{b}{m_1} + \frac{a}{2} \quad \text{and} \quad y_2 = \frac{m_2^2}{2}$$
So the slope of the line between these points is:

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{m^4 - 2bm - am^2}{2m^3 - b}
\]
$x_1 = \frac{b}{2m^2}$

$y_1 = \frac{b}{m} + \frac{a}{2}$

$x_2 = m$

$y_2 = \frac{m^2}{2}$

So the slope of the line between these points is:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{m^4 - 2bm - am^2}{2m^3 - b}$

$m(m^3 + am + b) = 0$

$m^3 + am + b = 0$
Solutions for cubic polynomials

Real roots of $x^3 + ax + b$ are the slope of a simultaneous tangent to:

$$\left(y - \frac{1}{2}a\right)^2 = 2bx \quad \text{and} \quad y = \frac{1}{2}x^2$$
Example: \( a = 2 \) and \( b = 1 \)

\[
(y - 1)^2 = 2x \quad \text{and} \quad y = \frac{1}{2}x^2
\]

**Directrix**

\[
x = -\frac{1}{2}
\]

**Focus**

\[
\left(\frac{1}{2}, 1\right)
\]

\[
\left(0, \frac{1}{2}\right)
\]
\[(y - 1)^2 = 2x\] \[y = \frac{1}{2}x^2\]
The method

\[ P_1 = (0.5, 1), \, \ell_1 : x = -0.5 \quad P_2 = (0, 0.5), \, \ell_2 : y = -0.5 \]

1. **Construct the x and y axis.**
2. Identify \( P_1, \, P_2, \, \ell_1 \) and \( \ell_2 \) in the paper.
3. Make a fold such that \( P_1 \) touches \( \ell_1 \) and \( P_1 \) touches \( \ell_1 \) at the same time. The slope \( m \) of the resulting line is the solution.
4. Find the point \((m, 0)\)
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1. Construct the \( x \) and \( y \) axis.
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\[(y - 1)^2 = 2x\] \[y = \frac{1}{2}x^2\] 

\[y \approx -0.4534x - 0.102786\]
Remark

The real solution of $x^3 + 2x + 1$ is not rational:

$$x = \sqrt[3]{\frac{1}{2}} \left( \sqrt[3]{177} - 9 \right) - 2 \sqrt[3]{\frac{2}{3 \left( \sqrt[3]{177} - 9 \right)}}$$
Huzita Axioms

1. Given two points $P_1$ and $P_2$ there is a unique fold passing through both of them.

2. Given two points $P_1$ and $P_2$ there is a unique fold placing $P_1$ onto $P_2$.

3. Given two lines $L_1$ and $L_2$, there is a fold placing $L_1$ onto $L_2$.

4. Given a point $P$ and a line $L$, there is a unique fold perpendicular to $L$ passing through $P$.

5. Given two points $P_1$ and $P_2$ and a line $L$, there is a fold placing $P_1$ onto $L$ and passing through $P_2$.

6. Given two points $P_1$ and $P_2$ and two lines $L_1$ and $L_2$, there is a fold placing $P_1$ onto $L_1$ and $P_2$ onto $L_2$.

7. Given a point $P$ and two lines $L_1$ and $L_2$, there is a fold placing $P$ onto $L_1$ and perpendicular to $L_2$. 

Let $O$ be the set of numbers that are constructible using **origami**.

$A$ is the set of numbers that are constructible with **ruler and compass**.
Let $\mathcal{O}$ be the set of numbers that are constructible using **origami**.

$\mathcal{A}$ is the set of numbers that are constructible with **ruler and compass**.

$\mathcal{A} \subsetneq \mathcal{O}$
Origami numbers

Let $\mathcal{O}$ be the set of numbers that are constructible using origami.

\[
\alpha \in \mathcal{O} \iff \alpha \text{ is constructible by marked ruler}
\]

\[
\iff \alpha \text{ is constructible by intersecting conics}
\]

\[
\iff \alpha \text{ lies on a 2-3 tower } \mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n
\]

\[
\iff \alpha \text{ is algebraic over } \mathbb{Q} \text{ with minimal polynomial of degree } 2^k 3^l
\]
Thank you!