Towards Stable Radial Basis Function Methods for Linear Advection Problems
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Abstract
In this work, we investigate (energy) stability of radial basis function (RBF) methods for linear advection problems. Classically, boundary conditions (BC) are enforced strongly in RBF methods. By now it is well-known that this can lead to stability problems, however. Here, we follow a different path and propose two novel RBF approaches which are based on a weak enforcement of BCs. By using the concept of flux reconstruction and simultaneous approximation terms (SATs), respectively, we are able to prove that both new RBF schemes are strongly (energy) stable. Numerical results in one and two spatial dimensions for both scalar equations and systems are presented, supporting our theoretical analysis.

Keywords: Linear advection, radial basis function methods, energy stability, flux reconstruction, simultaneous approximation terms

1. Introduction
Since their introduction in Hardy’s work \cite{hardy1971} on cartography in 1971, RBFs have become a powerful tool in numerical analysis, including multivariate interpolation and approximation theory. This is because they are, for instance, easy to implement, allow arbitrary scattered data, and can be highly accurate. Hence, RBFs are also often used in the numerical treatment of partial differential equations (PDEs) \cite{wells1990, wang1998, wu1998, zhou1999, zhou2000, xu2002} and are considered as a viable alternative to more traditional methods such as finite difference (FD), finite volume (FV), finite element (FE), and spectral schemes. In this work, we focus on RBF methods for linear advection problems of the form
\begin{equation}
\frac{\partial}{\partial t} \mathbf{u} + \sum_{r=1}^{d} A_r \partial_{x_r} \mathbf{u} = 0, \quad \mathbf{x} = (x_1, \ldots, x_d) \in \Omega \subset \mathbb{R}^d, \ t > 0,
\end{equation}
subject to appropriate boundary and initial conditions (ICs); see §2 for more details. Here, the vector of unknowns \( \mathbf{u} = (u_1, \ldots, u_m)^T \) contains the conserved variables. Note that under certain assumptions on the matrices \( A_r \in \mathbb{R}^{m \times m}, \ r = 1, \ldots, d \), this setting yields the subclass of hyperbolic conservation laws with linear flux functions \cite{leveque1992, tadmor1994, zhang1994}.

1.1. State of the Art
Even though RBF methods have a long-standing history in the context of numerical PDEs, their stability theory can still be considered as under-developed, especially when compared to more traditional methods. For instance, to the best of our knowledge, energy stability for RBF methods has only been considered in

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the recent works [26] and [23, Chapter 7.2]. There, it was shown that usual RBF methods — for which BCs are enforced strongly — are not energy stable and might therefore produce physically unreasonable numerical solutions. Furthermore, it appears that the differentiation matrices of RBF methods encountered in time-dependent PDEs often have eigenvalues with a positive real part resulting in unstable methods; see [57]. Hence, in the presence of rounding errors, these methods are less accurate [43, 56, 61] and can become unstable in time unless a dissipative time integration method [52, 57], artificial dissipation [15, 27, 29, 55, 58], or some other stabilizing technique [11, 20, 22, 25, 30, 37, 60] is used. So far, this issue was only overcome for problems which are free of BCs [52]. Especially, it was proven in [52] that RBF collocation methods are time-stable (in the sense of eigenvalues for linear problems) for all conditionally positive definite RBFs and node distributions when no BCs are present. RBF methods might therefore be well suited for periodic domains, such as circles and spheres, but not for applications with inflow-outflow conditions or general initial-boundary-value (IBV) problems where periodicity of the computational domain Ω cannot be assumed.

1.2. Our Contribution

The present work strives to establish stable RBF methods also in the presence of BCs. In particular, we propose and investigate two new approaches that involve a weak enforcement of BCs. These approaches yield two novel RBF schemes which are provably energy stable for linear advection equations (1). The first method draws inspiration from flux reconstruction (FR) schemes [41], a subclass of spectral element (SE) methods, and is subsequently referred to as the FR-RBF method. The second method, on the other hand, is inspired by SATs. These originate from FD methods [33, 34, 46] and, together with summation by parts (SBP) operators [14, 63], can be used to construct energy stable methods for many problems. Recently, this technique has been demonstrated to be viable also in the context of FE methods [3, 4]. Here, we adapt this approach to construct energy stable RBF methods for linear advection equations. Henceforth, this second class of energy stable RBF methods will be called SAT-RBF methods. Both methods are thoroughly investigated — theoretically as well as numerically — and their advantages but also pitfalls are highlighted. In our opinion, these methods should not be considered as some kind of ‘ultimate’ RBF schemes for linear advection equations. Rather, we hope that the present investigation will pave the way towards the development of a more mature stability theory for RBF methods. Moreover, the present work reveals some new connections between RBF methods and classical FD and FE schemes.

1.3. Outline

The rest of this manuscript is organized as follows. In §2, we detail the problem statement and main goal of this work, which is the construction of energy stable RBF methods for linear advection equations. §3 then provides a short recap of all necessary preliminaries on RBF methods. The heart of this work are §4 and §5, where we introduce and analyze two new approaches to weakly enforce BCs in RBF schemes. This results in the FR-RBF and SAT-RBF method. Both schemes are proven to be conservative and energy stable for linear advection equations. Yet, it should be pointed out that problems related to numerical stability arise in the construction of the FR-RBF method, as it can already be observed for scalar linear advection problems in one spatial dimension. We therefore only address the extension of the SAT-RBF method to multiple spatial dimensions and systems. Finally, §6 provides numerical results for both methods, along with a comparison with the usual RBF method (where the BCs are enforced strongly). The tests are performed for scalar equations and systems in one and two spatial dimensions. Concluding thoughts and some final remarks are offered in §7.
2. Problem Statement

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial \Omega$. We consider linear advection equations of the form (1) equipped with suitable IC and BC:

$$\partial_t u + \sum_{r=1}^{d} A_r \partial_{x_r} u = 0, \quad x \in \Omega, \ t > 0,$$

$$u = u_{\text{init}}, \quad x \in \Omega, \ t = 0,$$

$$u = g, \quad x \in \partial \Omega, \ t > 0,$$

where $u_{\text{init}}$ denotes the IC and $g$ describes the BC. Following [53], the conditions on $u_{\text{init}}$ and $g$ are chosen such that (2) is well-posed. For ease of notation, we focus on the one-dimensional case with positive constant velocity $A_1 = a > 0$ for the moment. Then, (2) reduces to

$$\partial_t u + a \partial_x u = 0, \quad x_L < x < x_R, \ t > 0,$$

$$u(x,0) = u_{\text{init}}(x), \quad x_L \leq x \leq x_R,$$

$$u(x_L,t) = g(t), \quad t > 0,$$

(3)

Note that in this case the BC is only defined at the left boundary. In general, the BC will be enforced at the inflow part of the boundary, denoted by $\partial \Omega^-$. As it is described in §3, the spatial and temporal discretization of (3) is often decoupled, yielding an ordinary differential equation (ODE), called the semidiscrete equation, which is then integrated in time. Similarly to discretizations used in the context of FE methods, for a fixed time $t$, the function $u(\cdot, t) : [x_L, x_R] \to \mathbb{R}$ is approximated by a suitable combination of basis functions $\varphi_1, \ldots, \varphi_N$ whose coefficients are to be computed.

Let us denote this semidiscretization at a fixed time $t$ by $u_N(t) : [x_L, x_R] \to \mathbb{R}$. Thus, $u(x,t) \approx u_N(x,t)$ for all $x \in [x_L, x_R]$ and fixed $t \geq 0$ and (3) becomes

$$\partial_t u_N + a \partial_x u_N = 0, \quad x_L < x < x_R, \ t > 0,$$

$$u_N(x,0) = (u_{\text{init}})_N(x), \quad x_L \leq x \leq x_R,$$

$$u_N(x_L,t) = g(t), \quad t > 0,$$

(4)

where $(u_{\text{init}})_N$ denotes is the approximation of the initial condition $u_{\text{init}}$. In this context, stability of a semidiscretization is usually defined as follows; see [53].

**Definition 1.** The semidiscrete method (4) is called strongly energy stable if

$$\|u_N(t)\|^2 \leq K(t) \left( \|(u_{\text{init}})_N\|^2 + \max_{s \in [0,t]} |g(s)|^2 \right)$$

(5)

holds for some $K(t)$ bounded for any finite $t$ and independent of $u_{\text{init}}, g$, and $N$ (the grid’s resolution). Furthermore, $\|\cdot\|$ is the norm corresponding to some inner product on the linear space $V_N$ spanned by the basis functions; that is, $V_N = \text{span}\{\varphi_1, \ldots, \varphi_N\}$.

Here, we consider the $L^2$ norm, given by

$$\|f\|^2 = \int_{\Omega} f(x) \, dx, \quad f \in L^2(\Omega),$$

(6)

which is a usual choice for $\|\cdot\|$. Other often considered norms include certain Sobolev [65] and discrete norms [19, 28, 54, 58, 59]. In the context of RBF methods, appropriate discrete norms would correspond to stable high-order quadrature/cubature rules for (potentially) scattered data points [24, 40]. For nonlinear problems, sometimes also the total variation is considered; see [8, 9, 22, 36, 64] and references therein.
Remark 2. Definition 1 is adapted to problem (3) where only one boundary term is fixed. However, if an additional forcing function was considered on the right hand side of (3), we would need to include the maximum of this function in (5) in the same fashion as for \( g \); see [63] for more details.

In particular, strong energy stability ensures that the numerical solution is not unconditionally increasing. It is therefore one of the basic requirements a method should fulfill. In §4 and §5, we propose and investigate two new approaches to ensure strong energy stability for RBF methods.

3. Preliminaries on Radial Basis Function Methods

In this section, we collect all the necessary preliminaries regarding RBF methods. For more details, we recommend the monographs [7, 13, 66] as well as the reviews [18, 42].

3.1. Method of Lines

Many RBF methods for time-dependent PDEs build up on the method of lines [50]. In this approach, problem (1) initially remains continuous in time and only a spatial discretization is considered. This results in a system of ODEs

\[ \frac{d}{dt} u_j = L(u)_j, \quad j = 1, \ldots, m, \]  

usually referred to as the semidiscrete equation. Here, \( u \) is given by \( u = (u_1, \ldots, u_m) \), where \( u_j = u_j(t) \) denotes the vector of coefficients of the spatial semidiscretization \( (u_j)_N \) of the \( j \)th component \( u_j = u_j(t, x) \).

Furthermore, \( L \) represents the operator for the spatial semidiscretization of the right hand side of (1). Once such a spatial semidiscretization has been defined, the semidiscrete equation (7) is evolved in time by some usual time integration method. A popular choice with favorable stability properties are strong stability preserving (SSP) Runge–Kutta (RK) methods [31, 32, 45, 51, 62]. For all numerical tests presented in this work, we used the explicit SSP-RK method of third order using three stages (SSPRK(3,3)), given as follows.

**Definition 3 (SSPRK(3,3)).** Let \( u^n \) be the solution at time \( t^n \). The solution \( u^{n+1} \) at time \( t^{n+1} \) is computed as

\[
\begin{align*}
  u^{(1)} &= u^n + \Delta t L(u^n), \\
  u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L\left(u^{(1)}\right), \\
  u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L\left(u^{(2)}\right).
\end{align*}
\]

The time step size \( \Delta t \) in (8) is computed as \( \Delta t = \frac{Ch}{\lambda_{\max}} \) with \( C = 0.1 \), where \( \lambda_{\max} \) denotes the largest characteristic velocity. Moreover, \( h \) is the smallest distance between any two distinct centers. That is, \( h = \min_{i \neq j} \|x_i - x_j\|_2 \). See §3.2 for more details.

For the general linear advection equation (1), the largest characteristic velocity \( \lambda_{\max} \) is given by the largest absolute eigenvalue among all matrices \( A_1, \ldots, A_d \).

3.2. Radial Basis Function Interpolation

Let us now consider the approximation of a function \( u : \Omega \to \mathbb{R} \) with \( \Omega \subset \mathbb{R}^d \) by RBF interpolants. Given a set of \( N \) distinct points \( x_n \in \Omega \), \( n = 1, \ldots, N \), called centers (or nodes), the RBF interpolant of \( u \) w. r. t. \( \varphi \) is given by

\[ u_N(x) = \sum_{n=1}^{N} \alpha_n \varphi \left( \|x - x_n\|_2 \right), \]

where \( \varphi : [0, \infty) \to \mathbb{R} \) only depends on the norm (radius) \( r = \|x - x_n\|_2 \) and is therefore referred to as an RBF (or kernel). The coefficients \( \alpha_n \), \( n = 1, \ldots, N \), are then uniquely determined by the interpolation conditions

\[ u_N(x_n) = u(x_n), \quad n = 1, \ldots, N. \]
These yield a system of linear equations,

\[
\begin{pmatrix}
\varphi (\|x_1 - x_1\|_2) & \ldots & \varphi (\|x_1 - x_N\|_2) \\
\vdots & \ddots & \vdots \\
\varphi (\|x_N - x_1\|_2) & \ldots & \varphi (\|x_N - x_N\|_2)
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_N
\end{pmatrix}
=:
\begin{pmatrix}
u(x_1) \\
\vdots \\
u(x_N)
\end{pmatrix},
\]

which can be solved for the vector of coefficients \( \alpha \in \mathbb{R}^N \) if the matrix \( \Phi \) is invertible. Some popular examples of RBFs are listed in Table 1 and even more can be found in the literature [7, 13, 18, 42, 66].

<table>
<thead>
<tr>
<th>RBF</th>
<th>( \varphi(r) )</th>
<th>parameter</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>( \exp(-\varepsilon r^2) )</td>
<td>( \varepsilon &gt; 0 )</td>
<td>0</td>
</tr>
<tr>
<td>Multiquadrics</td>
<td>( \sqrt{\varepsilon r^2 + 1} )</td>
<td>( \varepsilon &gt; 0 )</td>
<td>1</td>
</tr>
<tr>
<td>Polyharmonic splines</td>
<td>( r^{2k-1} )</td>
<td>( k \in \mathbb{N} )</td>
<td>( k )</td>
</tr>
<tr>
<td></td>
<td>( r^{2k} \log r )</td>
<td>( k \in \mathbb{N} )</td>
<td>( k+1 )</td>
</tr>
</tbody>
</table>

Table 1: Some popular RBFs

In many situations the RBF interpolant (9) is desired to include polynomials up to a certain degree, together with matching constraints on the expansion coefficients. Let \( \{p_i\}_{i=1}^Q \) be a basis of the space of (algebraic) polynomials of degree less than \( m \), denoted by \( \mathbb{P}_{m-1}(\mathbb{R}^d) \). Then, including polynomials of degree less than \( m \) into the RBF interpolant (9) results in

\[
u_N(x) = \sum_{n=1}^N \alpha_n \varphi (\|x - x_n\|_2) + \sum_{i=1}^Q \beta_i p_i(x) .
\]

Moreover, the matching constraints

\[
\sum_{n=1}^N \alpha_n p_i(x_n) = 0, \quad i = 1, \ldots, Q,
\]

have to be satisfied. These are supposed to ensure that (12) is still uniquely determined by the interpolation conditions (10). Let us denote

\[
P = \begin{pmatrix}
p_1(x_1) & \ldots & p_1(x_N) \\
\vdots & \ddots & \vdots \\
p_Q(x_1) & \ldots & p_Q(x_N)
\end{pmatrix}, \quad \beta = \begin{pmatrix}
\beta_1 \\
\vdots \\
\beta_Q
\end{pmatrix}.
\]

Then, (13) can be rewritten as \( P\alpha = 0 \) and the whole set of the expansion coefficients \( \alpha_1, \ldots, \alpha_N \) and \( \beta_1, \ldots, \beta_Q \) in (12) can be recovered from the system of linear equations

\[
\begin{pmatrix}
\Phi & P^T \\
P & 0
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
=:
\begin{pmatrix}
u \\
0
\end{pmatrix},
\]

where \( V \) is referred to as the Vandermonde matrix. There are various good reasons for including polynomials in RBF interpolants [6, 15, 16, 61]:

\footnote{Note that \( Q \) denotes the dimension of \( \mathbb{P}_{m-1}(\mathbb{R}^d) \), given by \( Q = \binom{d+m-1}{d} \).}
1. Polynomial terms can ensure that (15) is uniquely solvable when conditionally positive definite RBFs are used, assuming that the set of centers \( X := \{x_n\}_{n=1}^N \) is \( \mathbb{P}_{m-1}(\mathbb{R}^d) \)-unisolvent.\(^2\) See, for instance, [13, Chapter 7] or [23, Chapter 3.1].

2. Numerical tests demonstrate that including a constant improves the accuracy of derivative approximations. In particular, adding a constant avoids oscillatory representations of constant functions.

3. Including polynomial terms of low order can also improve the accuracy of RBF interpolants near domain boundaries due to regularizing the far-field growth of RBF interpolants [17].

Finally, we note that the set of all RBF interpolants (12) forms an \( N \)-dimensional linear space, denoted by \( V_{N,m} \). This space is spanned, for instance, by the basis elements

\[
\psi_k(x) = \sum_{n=1}^N \alpha_{kn}^{(k)} \varphi(||x - x_n||_2) + \sum_{i=1}^Q \beta_{ik}^{(k)} p_i(x), \quad k = 1, \ldots, N,
\]

which (expansion coefficients) are uniquely determined by

\[
\psi_k(x_n) = \delta_{kn} := \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise}, \end{cases} \quad n = 1, \ldots, N,
\]

and the matching conditions (13). The basis \( \{\psi_k\}_{k=1}^N \) can be considered as a nodal basis and comes with the advantage of providing a representation of the RBF interpolant in which the expansion coefficients are simply given by the known nodal values of \( u \):

\[
u_N(x) = \sum_{k=1}^N u(x_k) \psi_k(x)
\]

This representation will be convenient for the implementation of the latter proposed RBF methods.

### 3.3. Usual Radial Basis Function Methods

For sake of simplicity, we only outline the procedure for usual RBF methods for a scalar linear advection equation of the form

\[
\partial_t u + \sum_{r=1}^d a_r \partial_{x_r} u = 0, \quad x \in \Omega \subset \mathbb{R}^d, \quad t > 0,
\]

\[
u = g, \quad x \in \partial \Omega, \quad t > 0,
\]

ignoring the IC for the moment. Yet, the extension to systems is straightforward and achieved by applying the procedure to every component of the system. Combining the method of lines from §3.1 with the above discussed RBF interpolants, the idea behind (collocation) RBF methods is to define the spatial discretization \( u \) as the values of \( u \) at a set of grid points and the operator \( L(u) \) by using the spatial derivative of the RBF interpolant \( u_N \). Let \( X = \{x_n\}_{n=1}^N \) be the set of grid points in the computational domain \( \Omega \subset \mathbb{R}^d \) and let us assume that some of these grid points lie at the boundary \( \partial \Omega \). In usual RBF methods, the BC is then enforced by replacing the original interpolation conditions (10) by

\[
u_N(x_n) = \begin{cases} u(x_n) & \text{if } x_n \notin \partial \Omega, \\ g(x_n) & \text{if } x_n \in \partial \Omega, \end{cases} \quad n = 1, \ldots, N.
\]

This is what is commonly referred to as a \textit{strong enforcement of BCs}. Finally, the usual RBF method can be summarized in three simple steps:

\(^2\)We say that an RBF \( \varphi \) is \textit{conditionally positive definite of order} \( m \) on \( \mathbb{R}^d \) if \( \alpha^T \Phi \alpha > 0 \) holds for every set of distinct centers \( X \subset \mathbb{R}^d \) and all \( \alpha \in \mathbb{R}^N \setminus \{0\} \) that satisfy (13).
1. Determine the RBF interpolant $u_N \in V_{N,m}$ satisfying (20).

2. Define $L(u)$ in the semidiscrete equation (7) by

$$L(u) = \left( -\sum_{r=1}^{d} a_r (\partial_x, u_N) (x_n) \right)^N_{n=1} \cdot (21)$$

3. Use some time integration method, e.g. SSPRK(3,3), to evolve (7) in time.

Even though a strong enforcement of BCs, as described in (20), yields a pleasantly simple scheme, it is known that this approach can result in RBF methods that are unstable in time; see [26, 57]. In fact, this can already be observed for a simple scalar linear advection equation in one spatial dimension and is demonstrated, for instance, in [26, 57] and [23, Chapter 7.2] as well as by Figure 2 in §6.

4. Linear Stable RBF Methods I: Flux Reconstruction

In this section, we propose and investigate the first approach to construct RBF methods which are strongly energy stable for linear advection problems. Henceforth, we will refer to such RBF methods simply as linear stable RBF methods.

4.1. Basic Idea

For sake of simplicity, we start by considering the one-dimensional scalar linear advection equation

$$\partial_t u + \partial_x (au) = 0, \quad x_L < x < x_R, \quad t > 0,$$

on $\Omega = [x_L, x_R]$ with constant velocity $a > 0$ (the case $a < 0$ can be treated analogously). In the context of (nonlinear) hyperbolic conservation laws, $f(u) = au$ is usually referred to as the flux function. Let $X = \{x_n\}_{n=1}^{N}$ be a set of distinct centers in $\Omega$, $m \in \mathbb{N}$, and let $u_N$ and $f_N$ respectively denote the corresponding RBF interpolant of $u$ and $f(u)$ including polynomials of degree less than $m$; see (12). Note that $X$ does not necessarily include the boundary points $x_L$ and $x_R$. The idea behind this first approach to construct linear stable RBF methods is to introduce correction functions $c_L, c_R : \Omega \to \mathbb{R}$ and to consider the reconstructed flux function

$$f_N^{\text{rc}}(x) = f_N(x) + c_L(x) [f_N^{\text{num}} - f_N(x_L)] + c_R(x) [f_N^{\text{num}} - f_N(x_R)].$$

Here, $f_N^{\text{num}}$ and $f_R^{\text{num}}$ are the values of a numerical flux,

$$f_N^{\text{num}} = f_N^{\text{num}} (g_L(t), u_N(x_L)),$$

which is chosen to be consistent ($f_N^{\text{num}}(u, u) = f(u)$), Lipschitz continuous, and monotone. Examples of commonly used numerical fluxes can be found in [8] and [64]. For the linear advection equation (22) a usual choice is the upwind flux

$$f_N^{\text{num}}(u, v) = \begin{cases} 
au & \text{if } a \geq 0, \\
au & \text{if } a < 0.
\end{cases}$$

Substituting the solution $u$ by $u_N$ and the flux function $au$ by (23), at a fixed time $t$, we end up with the spatial semidiscretization

$$\partial_t u_N = -f'_N - c_L \left[ f_N^{\text{num}} - f_N(x_L) \right] - c_R \left[ f_N^{\text{num}} - f_N(x_R) \right].$$

Henceforth, we call this semidiscretization and the resulting numerical scheme the flux reconstruction RBF (FR-RBF) method. It should be noted that the idea behind the FR-RBF method stems from FR schemes [41] which were first introduced by Huygh in 2007 in the context of polynomial FE methods. In what follows, we show how certain conditions for the correction functions $c_L, c_R$ yield provable conservative and linear stable RBF methods. Furthermore, we will address the construction of these correction functions.
4.2. Conservation and Linear Stability

A fundamental property of linear advection equations — and general hyperbolic conservation laws — is conservation. That is, the rate of change of the total amount of the conserved variable $u$ should be equal to the flux across the domain boundaries. Any numerical method should satisfy this property on a discrete level; see [49] and references therein. For the above proposed FR-RBF method, indeed, this can be ensured by requiring the correction functions $c_L, c_R$ to satisfy

$$c_L(x_R) = c_R(x_L) = 0, \quad c_R(x_R) = c_L(x_L) = 1.$$  \hspace{1cm} (27)

This is summarized and proven in the following lemma.

**Lemma 4** (Conservation of the FR-RBF method). Given is the FR-RBF method (26) and correction functions satisfying (27). Then, the FR-RBF method is conservative; that is,

$$\frac{d}{dt} \int_{\Omega} u_N \, dx = -(f_{R}^{\text{num}} - f_{L}^{\text{num}})$$  \hspace{1cm} (28)

holds for all times $t$.

**Proof.** Integrating (26) over $\Omega$, we get

$$\frac{d}{dt} \int_{\Omega} u_N \, dx = - \int_{\Omega} f'_N \, dx - \left[ f_{L}^{\text{num}} - f_{N}(x_L) \right] \int_{\Omega} c'_L \, dx - \left[ f_{R}^{\text{num}} - f_{N}(x_R) \right] \int_{\Omega} c'_R \, dx$$

$$= - \left[ f_{N}(x_R) - f_{N}(x_L) \right] - \left[ f_{L}^{\text{num}} - f_{N}(x_L) \right] \left[ c_L(x_R) - c_L(x_L) \right]$$

$$- \left[ f_{R}^{\text{num}} - f_{N}(x_R) \right] \left[ c_R(x_R) - c_R(x_L) \right].$$  \hspace{1cm} (29)

Since the correction functions satisfy (27), this yields

$$\frac{d}{dt} \int_{\Omega} u_N \, dx = - \left[ f_{N}(x_R) - f_{N}(x_L) \right] + \left[ f_{L}^{\text{num}} - f_{N}(x_L) \right] - \left[ f_{R}^{\text{num}} - f_{N}(x_R) \right]$$

$$= f_{L}^{\text{num}} - f_{R}^{\text{num}},$$  \hspace{1cm} (30)

and therefore the assertion. \hfill $\square$

In a similar manner, linear stability can be ensured for the FR-RBF method by a clever choice of the correction functions $c_L, c_R$. This time, we need $c_L, c_R$ to satisfy

$$\int_{\Omega} v c'_L \, dx = -v(x_L), \quad \int_{\Omega} v c'_R \, dx = v(x_R)$$  \hspace{1cm} (31)

for all $v \in V_{N,m}$. Remember that $V_{N,m}$ denotes the space of all RBF interpolants (12), from which also the approximations $u_N$ and $f_N$ are chosen.

**Theorem 5** (Linear stability of the FR-RBF method). Given is the FR-RBF method (26) with correction functions satisfying (31) and the upwind flux $f_{\text{num}}$ as in (25). Then, the FR-RBF method is strongly energy stable for the linear advection equation (22); that is, (5) holds.

**Proof.** In case of the linear advection equation (22), we have $f_N = au_N$. Hence, we can note that for the $L^2$-norm

$$\frac{d}{dt} \| u_N(t) \|^2 = 2 \int_{\Omega} u_N (\partial_t u_N) \, dx$$

$$= - 2a \int_{\Omega} u_N u'_N \, dx - 2 \left[ f_{L}^{\text{num}} - au_N(x_L) \right] \int_{\Omega} u_N c'_L \, dx$$

$$- 2 \left[ f_{R}^{\text{num}} - au_N(x_R) \right] \int_{\Omega} u_N c'_R \, dx$$  \hspace{1cm} (32)
holds. Thus, when $c_L$ and $c_R$ satisfy the constraint (31), we get

$$\frac{d}{dt} \| u_N(t) \|^2 = -a \left[ u_N(x_R)^2 - u_N(x_L)^2 \right] + 2 \left[ f^\text{num}_L - a u_N(x_L) \right] u_N(x_L) - 2 \left[ f^\text{num}_R - a u_N(x_R) \right] u_N(x_R).$$

(33)

Furthermore, note that $f^\text{num}_L$ and $f^\text{num}_R$ are given by the upwind flux (25) as

$$f^\text{num}_L = f^\text{num}(g(t), u_N(x_L)) = a g(t),$$

$$f^\text{num}_R = f^\text{num}(u_N(x_R), u_N(x_R)) = a u_N(x_R),$$

(34)

since the BC only applies to the left boundary and there is no BC at the right boundary. Finally, this yields

$$\frac{d}{dt} \| u_N(t) \|^2 \leq -au_N(x_L)^2 + 2au_N(x_L)g(t)$$

$$= -a \left[ u_N(x_L) - g(t) \right]^2 + ag(t)^2$$

$$\leq ag(t)^2$$

(35)

and therefore

$$\| u_N(t) \|^2 \leq \| (u_{\text{init}})_N \|^2 + a \int_0^t g(s)^2 \, ds$$

$$\leq K(t) \left( \| (u_{\text{init}})_N \|^2 + \max_{s \in [0,t]} |g(s)|^2 \right)$$

(36)

with $K(t) = \max\{1, a t\}$. \hfill \Box

4.3. On the Construction of $c_L$ and $c_R$

Following Lemma 4 and Theorem 5, the proposed FR-RBF method was proven to be conservative and linear stable if $c_L, c_R$ satisfy the conditions (27) and (31). Here, we show how such correction functions can be constructed. Thereby, we only address the construction of $c_L$ and note that $c_R$ can be constructed completely analogously. Collecting the conditions (27) and (31), $c_L$ should satisfy

$$c_L(x_L) = 1, \quad c_L(x_R) = 0,$$

(37)

$$\int_\Omega v \cdot c'_L \, dx = -\varphi(x_L) \quad \forall v \in V_{N,m}.$$  

(38)

In §3.2 we have already noted that $V_{N,m}$ is spanned by the nodal basis $\{\psi_k\}_{k=1}^N$ given in (16). Hence, condition (38) is equivalent to

$$\int_\Omega \psi_k c'_L \, dx = -\psi_k(x_L), \quad k = 1, \ldots, N.$$  

(39)

Together, the conditions (37) and (39) therefore impose $N + 2$ conditions on the correction function $c_L$. It should be stressed that, so far, we have not restricted $c_L$ to belong to any specific function space. For example in FR methods, which are the motivation for the FR-RBF method, $c_L$ lies in certain polynomial spaces. Regarding the FR-RBF method, however, it seems natural to let $c_L$ be an RBF interpolant.\(^3\) Thus, we follow the approach of $c_L$ being an RBF interpolant using $N + 2$ centers in $\Omega$ and including polynomials of degree less than $m$; that is, $c_L \in V_{N+2,m}$. Theoretically, the corresponding set of centers $\tilde{X} = \{\tilde{x}_n\}_{n=0}^{N+1}$ can be chosen completely independent of the original set of centers $X$. In our numerical tests we always chose

\(^3\)Of course, other choices are possible as well. Yet, in this work, we will only consider correction functions which are RBF interpolants themselves.
and \( \tilde{X} \) respectively to be the set of \( N \) and \( N + 2 \) equidistant points in \([x_L, x_R]\) including the boundary points. Next, letting \( \{\tilde{\psi}_j\}_{j=0}^{N+1} \) be a basis of \( V_{N+2,m} \), \( c_L \) can be represented as

\[
c_L(x) = \sum_{j=0}^{N+1} \gamma_j \tilde{\psi}_j(x).
\]

(40)

Then, the conditions (37) and (39) become

\[
\sum_{j=0}^{N+1} \gamma_j \tilde{\psi}_j(x_L) = 1, \quad \sum_{j=0}^{N+1} \gamma_j \tilde{\psi}_j(x_R) = 0,
\]

(41)

\[
\sum_{j=0}^{N+1} \gamma_j \langle \psi_k, \tilde{\psi}_j' \rangle = -\psi_k(x_L), \quad k = 1, \ldots, N,
\]

(42)

where

\[
\langle \psi_k, \tilde{\psi}_j' \rangle = \int_{\Omega} \psi_k \tilde{\psi}_j' dx.
\]

(43)

Hence, the coefficients \( \gamma_j \) can be determined by solving the systems of linear equations

\[
\begin{pmatrix}
\langle \tilde{\psi}_0(x_L), \psi_1 \rangle & \cdots & \langle \tilde{\psi}_0(x_L), \psi_{N+1} \rangle \\
\langle \tilde{\psi}_0(x_R), \psi_1 \rangle & \cdots & \langle \tilde{\psi}_0(x_R), \psi_{N+1} \rangle \\
\langle \psi_1, \psi_0 \rangle & \cdots & \langle \psi_1, \psi_{N+1} \rangle \\
\vdots & \ddots & \vdots \\
\langle \psi_N, \psi_0 \rangle & \cdots & \langle \psi_N, \psi_{N+1} \rangle \\
\end{pmatrix}
\begin{pmatrix}
\gamma_0 \\
\gamma_1 \\
\vdots \\
\gamma_N+1 \\
\end{pmatrix}
=:
\begin{pmatrix}
1 \\
0 \\
\vdots \\
-\psi_1(x_L) \\
\end{pmatrix}.
\]

(44)

Substituting these coefficients into (40) we can also recover the correction function \( c_L \).

4.4. Advantages and Pitfalls

An obvious advantage of the FR-RBF method discussed above is that it is provably conservative and linear stable. For usual RBF methods this is not the case, which is demonstrated in §6. Hence, unlike usual RBF methods, the FR-RBF method is ensured to be stable and produce physically reasonable solutions.

A disadvantage of this method lies in the construction of the correction functions \( c_L \) and \( c_R \), however. In particular, these can only be recovered from the system of linear equations (44) if the matrix \( A \) is regular. Note that the regularity of \( A \) depends on the kernel \( \varphi \), the sets \( X \) and \( \tilde{X} \), and the polynomial degree \( m - 1 \). Unfortunately, we do not see how the regularity of \( A \) can be ensured. Even though we observe \( A \) to be regular in all our numerical tests, we also note that it often has a fairly high condition number. This is reported in Table 2 for the cubic and quintic kernel on an increasing set of equidistant points.

<table>
<thead>
<tr>
<th></th>
<th>cubic kernel ( \varphi )</th>
<th>quintic kernel ( \varphi )</th>
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</thead>
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<td>( A ) regular?</td>
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<td>\checkmark</td>
</tr>
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<td>4.0E+10</td>
<td>5.4E+12</td>
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</table>

Table 2: Condition numbers of matrix \( A \) in (44) for the cubic and quintic kernel.

We can note from Table 2 that the proposed construction of \( c_L \) and \( c_R \) can be expected to result in a numerically unstable — or at least only pseudo stable — method. This is also demonstrated in the later numerical tests.
5. Linear Stable RBF Methods II: Simultaneous Approximation Terms

In the previous section, the FR approach was proposed to construct linear stable RBF methods. Unfortunately, we also noted some problems for this approach related to numerical instabilities in the construction of the correction functions $c_L, c_R$. In this section, we therefore follow a different approach, considering SATs in combination with a RBF method. By now, SATs are an often used tool in the context of FD methods. Together with SBP operators they are applied to retrieve energy stable (SBP-SAT) schemes; see [14, 63] and references therein. Recently, the SAT approach has also been combined with continuous FE methods; see [3, 4, 38]. Here, we propose to use SATs in combination with RBF methods for determining boundary correction operators that yield provable linear stability. To the best of our knowledge, this is the first time SATs are investigated in the context of RBF methods.

5.1. Basic Idea in One Dimension

The basic idea of the SAT approach is to weakly implement the BCs in the spatial semidiscretization. For a scalar equation in one dimension, this yields the spatial semidiscretization

$$\partial_t u_N = - f_N + SAT_L + SAT_R,$$

where $SAT_L$ and $SAT_R$ represent the SATs. The SATs are defined on the inflow part of the boundary through

$$SAT_L = \tau_L a^+ \delta_{x_L} (u_N - g),$$
$$SAT_R = \tau_R a^- \delta_{x_R} (u_N - g),$$

(46)

where $a^+ = \max\{a, 0\}$ and $a^- = \min\{a, 0\}$. The terms $\tau_L$ and $\tau_R$ denote the boundary operators, while $\delta$ is the Dirac delta. That is, the generalized function with the property that

$$\int_{\Omega} v(x) \delta_{x_L} (x) \, dx = v(x_L).$$

(47)

The boundary operators can be seen as some penalty terms and the basic idea of the SAT approach is to weakly impose the BCs. Regarding linear stability of (45), let us consider the linear advection equation (22) with constant velocity $a > 0$. An inflow condition is set on the left boundary while an outflow is prescribed on the right boundary. Hence, the only important boundary term in the semidiscretization (45) is $SAT_L$. In this case, requiring $\tau_L < -\frac{1}{2}$ already suffices to ensure linear stability.

**Lemma 6** (Linear stability of the SAT-RBF method). Given is the SAT-RBF method (45) with SATs (46) and $\tau_L < -\frac{1}{2}$. Then, the SAT-RBF method is strongly energy stable for the linear advection equation (22); that is, (5) holds.

**Proof.** From (45), we get the conservation relation:

$$\frac{d}{dt} \int_{\Omega} u_N^2 \, dx = -2a \int_{\Omega} u_N u_N' \, dx + 2u_N SAT_L$$

$$= -a \left[ u_N(x_R)^2 - u_N(x_L)^2 \right] + 2u_N(x_L)\tau_L a^+ (u_N(x_L) - g)$$

$$= -au_N(x_R)^2 + au_N(x_L)^2 + 2\tau_L au_N(x_L)(u_N(x_L) - g).$$

(48)

As obviously $au_N(x_R)^2 > 0$, we can bound the above equation by

$$\frac{d}{dt} \int_{\Omega} u_N^2 \, dx \leq au_N(x_L)^2 + 2\tau_L au_N(x_L)(u_N(x_L) - g).$$

(49)

Extending this equation under the condition $\tau_L < -\frac{1}{2}$ yields

$$\frac{d}{dt} \int_{\Omega} u_N^2 \, dx \leq (1 + 2\tau_L)au_N(x_L)^2 - 2\tau_L au_N(x_L) + \frac{a\tau_L^2}{1 + 2\tau_L} g^2 - \frac{a\tau_L^2}{1 + 2\tau_L} g^2$$

$$\leq -\frac{\tau_L^2 a g^2}{1 + 2\tau_L}.$$

(50)
which implies energy stability in accordance to Definition 1.

Remark 7. The above described SAT approach can be interpreted as the use of a numerical flux which is determined to guarantee the energy estimate. Hence, regarding the results obtained in §4, the correction terms in the FR approach can be interpreted as SATs, and vice-versa.

Remark 8. Here, we only consider stability for continuous $L^2$ norms, rather than discrete $L^2$ norms that are commonly considered in the SBP-SAT framework. In particular, this requires us to assume exact integration of the volume and the SAT terms. In general, considering arbitrary multidimensional domains, this can be a fairly strong numerical constraint. In case of linear problems with constant coefficients, the impact of this limitation is damped by the use of exact quadrature rules in the discretization, and by the existence of a SBP-like condition when using RBFs. However, this also opens the possibility for future research by determining point distributions and RBFs that satisfy an SBP property (at least up to machine precision). This work can therefore be seen as an attempt to build a bridge between these different perspectives on numerical PDEs. Furthermore, future works could address the extension of the proposed methods to nonlinear problems. For these, for instance, entropy correction as suggested in [1, 2, 4, 5] might be of interest.

5.2. Extension to Two Dimensional Problems

Next, we aim to extend SAT-RBF methods to multidimensional problems. As the extension to multidimensional problems can be done in the same way, for sake of simplicity, we only focus on two-dimensional model problems. Hence, let us consider the linear problem

\[
\partial_t u + a_1 \partial_x u + a_2 \partial_y u = 0, \quad x \in \Omega, \ t > 0, \\
u = u_{\text{init}}, \quad x \in \Omega, \ t = 0, \\
u = g, \quad x \in \partial \Omega, \ t > 0,
\]  

with $\Omega \subset \mathbb{R}^2$. By $\partial \Omega^-$ and $\partial \Omega^+$ we respectively denote the part of the boundary where inflow and outflow conditions are applied. Moreover, we assume that the imposed BCs yield to the well-posedeness of (51); see [53]. Discretizing (51) by the RBF-SATs approach yields

\[
\partial_t u_N = -a_1 \partial_x u_N - a_2 \partial_y u_N + SAT^\text{in}.
\]  

Here, $SAT^\text{in}$ represents the inflow BC. That is,

\[
SAT^\text{in} = \Pi \delta_{\partial \Omega^-} (u_N - g)
\]

with $\Pi = \Pi(x) \in \mathbb{R}^2$ being the boundary operator that remains to be characterized such that linear stability is guaranteed. Moreover, $\delta_{\partial \Omega^-}$ denotes the generalized function with the property that

\[
\int_{\Omega} \Pi \delta_{\partial \Omega^-} v \, dx = \int_{\partial \Omega^-} (\Pi \cdot n) v \, ds,
\]

where $n$ represents the outward normal to the boundary. In order to derive a stability estimate, we use the Gauss theorem to obtain

\[
\frac{d}{dt} \int_{\Omega} u_N^2 \, dx = - \int_{\partial \Omega^-} \left[ u_N^2 \left( (a_1, a_2)^T \cdot n \right) - 2u_N (\Pi \cdot n)(u_N - g) \right] \, ds.
\]

Thus, linear stability is guaranteed by the following admissibility condition on $\Pi$:

\[
u_N^2 \left( (a_1, a_2)^T \cdot n \right) - 2u_N (\Pi \cdot n)(u_N - g) \geq 0
\]

This expression highly depends on the domain and the BCs themselves. Assuming no incoming wave ($g|_{\partial \Omega^-} = 0$) condition (56) reduces to $2\Pi \cdot n \leq (a_1, a_2)^T \cdot n$. We summarize this in the following lemma.
Lemma 9 (Linear stability of the SAT-RBF method in two dimensions). Given is the SAT-RBF method (52) with boundary operator Π satisfying (56). Then, the SAT-RBF method is strongly energy stable for the two dimensional advection equation (51).

5.3. Extension to Linear Systems in One Space Dimension

Finally, we demonstrate how the SAT-RBF approach can be extended to systems of linear advection equations. Let us consider the following one-dimensional model problem:

\[
\begin{align*}
\partial_t u + A \partial_x u &= 0, \quad x \in (0, 1), \ t > 0, \\
L_0(u) &= g_0(t), \quad x = 0, \ t > 0, \\
L_1(u) &= g_1(t), \quad x = 1, \ t > 0, \\
u(x, 0) &= u_{\text{init}}, \quad x \in [0, 1], \ t = 0
\end{align*}
\] (57)

Here, \( u = (u_1, \ldots, u_m)^T \) is the vector of conserved variables and \( A \in \mathbb{R}^{m \times m} \) is a symmetric matrix. Furthermore, \( L_0 \) and \( L_1 \) are assumed to be linear operators. We denote by \( n_0 \) the number of incoming characteristics at \( x = 0 \) and by \( n_1 \) the number of incoming characteristics at \( x = 1 \). Thus, the rank of \( L_0 \) and \( L_1 \) respectively is \( n_0 \) and \( n_1 \). We start by noting that multiplying the continuous differential equation in (57) from the left hand side by \( u^T \), we get

\[
\int_0^1 u^T \partial_t u \, dx = - \int_0^1 u^T A \partial_x u \, dx.
\] (58)

As the energy is given by \( E = \frac{1}{2} \int_0^1 u^2 \, dx \), it satisfies through the use of the Gauss theorem

\[
\frac{d}{dt} E + u(1, t)^T A u(1, t) - u(0, t)^T A u(0, t) = 0.
\] (59)

We then apply the SAT term corresponding to the incoming characteristics. Imposing the BC weakly and using the RBF formulation for the problem (57) in (59), we obtain

\[
\frac{d}{dt} E_N = u_N(1, t)^T \left( -A u_N(1, t) + \Pi_1(L_1(u_N) - g_1) \right)
+ u_N(0, t)^T \left( A u_N(0, t) - \Pi_0(L_0(u_N) - g_0) \right),
\] (60)

where \( \Pi_0 \) and \( \Pi_0 \) are the boundary operators. These have to be chosen such that the change of energy yields an estimate as described in Definition 1. Moreover, for any \( t \), the image of the boundary operator \( L_0 \) and \( L_1 \) respectively has to be the same as the image of \( \Pi_0 L_0 \) and \( \Pi_1 L_1 \). This ensures that there is no loss of information at the boundary.

This can be solved by decomposing \( A \) in its diagonal form and rewriting the boundary procedure by using characteristic variables instead of \( u \). However, to specify precisely \( \Pi_0 \) and \( \Pi_1 \), one has to consider the problem itself, e.g. the wave equation. More details can be found in [3]. If we set the BC to zero at the inflow part and suppose that \( L_0 \) and \( L_1 \) are simply given by the identity matrix, we can derive from (60) the following result.

Lemma 10 (Linear stability of the SAT-RBF method for system). Assume that the boundary operator \( \Pi \) can be selected such that the matrix \( \Pi - A + (\Pi - A)^T \) is negative semi-definite. Then, the SAT-RBF method is strongly energy stable for the linear symmetric system (57).

6. Numerical Results

In what follows, we numerically investigate the two proposed approaches to construct linear stable RBF methods and compare them to the usual RBF method. All tests are performed for polyharmonic splines (PHS) using a cubic (\( k = 2 \)) and a quintic (\( k = 3 \)) kernel. Moreover, polynomials of degree less than \( m = 2 \)
and \( m = 3 \) have been included, respectively. This ensures that the resulting RBF interpolant is always well-defined; see [42] or [23, Chapter 3.1.5]. While the proposed RBF methods could also be applied to other kernels, such as Gaussian or compactly supported ones (Wendland functions), PHS allow us to circumvent a discussion on how the shape parameter \( \varepsilon \) should be chosen.\(^4\) Of course, a discussion and comparison of different kernels would be highly interesting. Yet, it would also exceed the scope of this work. The same holds for the distribution of the centers. It should be noted that the stability results discussed in §4 and §5 hold independently of the kernel \( \varphi \) and centers \( X \) (and \( \tilde{X} \)). Hence, for sake of simplicity, we only consider equidistant centers in the subsequent numerical tests.

6.1. One-Dimensional Scalar Problems

Let us start by considering the scalar one-dimensional advection problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(au) &= 0, & 0 < x < 1, \ t > 0, \\
u(t, x_L) &= g(t), & t > 0, \\
u(0, x) &= u_{\text{init}}(x), & 0 \leq x \leq 1, 
\end{align*}
\]

(61)

with constant velocity \( a = 1 \). For the SAT-RBF method we simply used \( \tau = -1 \) in all numerical tests.

6.1.1. Inflow Boundary Condition

Given are the BC and IC

\[
g(t) = u_{\text{init}}(0.5 - t), \quad u_{\text{init}}(x) = \begin{cases} 
\varepsilon^{16} \exp \left( \frac{-16}{1-(4x-1)^2} \right) & \text{if } 0 < x < 0.5, \\
0 & \text{otherwise}.
\end{cases}
\]

(62)

That is, we have a smooth IC and an inflow BC at the left boundary \( x = 0 \). The results for the usual RBF, FR-RBF and SAT-RBF method for \( N = 40 \) grid points at time \( t = 0.5 \) are displayed in Figure 1. From Figure 1, we can already note the numerical stability issues for the FR-RBF method which were discussed in §4.4. These stem from high condition numbers of the matrix \( A \) in the linear system (44) that has to be solved to recover the correction functions \( c_L \) and \( c_R \). In fact, this lack of numerical stability for the FR-RBF method can also be noted from increased errors in Table 3. Table 3 lists the \( \ell^1 \) and \( \ell^{\infty} \) errors,

\[
\| u - u_N \|_{\ell^1} = \frac{1}{N} \sum_{n=1}^{N} |u(x_n) - u_N(x_n)|, \\
\| u - u_N \|_{\ell^{\infty}} = \max_{n=1,\ldots,N} |u(x_n) - u_N(x_n)|,
\]

(63)

and the resulting average orders, \( O = \frac{1}{3} \sum_{j=1}^{3} o_j \), where

\[
o_j = \log_2 \left( \frac{\| u - u_{N_j+1} \|}{\| u - u_{N_j} \|} \right)
\]

(64)

with \( N_j = 10 \cdot 2^j \).

\(^4\)It is well-known that decreasing the shape parameter \( \varepsilon \) increases the accuracy of the RBF interpolant, while decreasing \( \varepsilon \) decreases the condition number of the Vandermonde matrix \( V \); see [13, Chapter 16] and references therein. One (non-optimal) strategy for choosing the shape parameter is therefore to select the smallest value of \( \varepsilon \) for which the obtained matrix can still be inverted numerically with a decent accuracy.
6.1.2. Long Time Simulation with Periodic Boundary Conditions

Next, let us consider (61) with periodic BCs and a smooth IC given by
\[ u(t, 0) = u(t, 1), \quad u_{\text{init}}(x) = \sin(\pi x). \]  
(65)

The results for \( N = 20 \) at time \( t = 10 \) are displayed in Figure 2. Moreover, Figure 2 illustrates the corresponding energy profiles of the different methods over time. Note that for this problem the energy of the solution should actually be constant over time. Yet, in both cases, using the cubic as well as the quintic kernel, we respectively observe the usual RBF method to result in clearly decreasing and increasing energy profiles. For the FR-RBF and SAT-RBF method, on the other hand, we observe this decrease/increase to be significantly smaller. Unfortunately, this test demonstrates once more the numerical instability of the FR-RBF method. While the FR-RBF method arguably provides the best energy profile for the cubic kernel, the method broke down for the quintic kernel before the final time could be reached. The SAT-RBF method, however, yielded stable computations in both cases. It should be noted that somewhat surprisingly the energy profile of the SAT-RBF method is also observed to slightly increase over time in Figure 2d. This seems to be a contradiction to our previous results that the SAT-RBF method is linear energy stable. Yet, we think that this might be caused by numerical rounding errors and time integration rather than by failure of the spatial semidiscretization. Still, the SAT-RBF method yields visibly more accurate energy profiles than the usual RBF method, which is also reflected in the quality of there numerical solution itself; see figures 2a and 2c. Furthermore, the \( \ell^1 \)- and \( \ell^\infty \)-erros as well as the corresponding average orders for this test problem can be found in Table 4.
6.2. Extension to Systems: The Wave Equation

Extending our investigation to one-dimensional systems of advection equations, let us now consider the wave equation

\[ \partial_{tt} u - c^2 \partial_{xx} u = 0, \quad 0 < x < 1, \quad t > 0 \]

(66)

with periodic BCs. Note that the wave equation can be rewritten as a system of linear advection equations,

\[
\begin{align*}
\partial_t u + c \partial_x v &= 0, \\
\partial_t v + c \partial_x u &= 0,
\end{align*}
\]

(67)

which is sometimes referred to as the one-dimensional acoustic problem; see [21, 28]. Its solution is given by

\[
\begin{align*}
u(t, x) &= \frac{1}{2} \left[ u_{\text{init}}(x - ct) + u_{\text{init}}(x + ct) \right] + \frac{1}{2} \left[ v_{\text{init}}(x - ct) - v_{\text{init}}(x + ct) \right], \\
v(t, x) &= \frac{1}{2} \left[ u_{\text{init}}(x - ct) - u_{\text{init}}(x + ct) \right] + \frac{1}{2} \left[ v_{\text{init}}(x - ct) + v_{\text{init}}(x + ct) \right].
\end{align*}
\]

(68)

Henceforth, we choose \( c = 1 \), ICs

\[ u_{\text{init}}(x) = e^{-20(2x-1)^2}, \quad v_{\text{init}}(x) = 0, \]

(69)

and consider the solution \((u, v)^T\) at time \( t = 1 \). Figure 3 illustrates these for the usual RBF and SAT-RBF method for \( N = 20 \). Furthermore, Table 5 lists the corresponding errors and average orders for an increasing number of centers. From this table we can note that the SAT-RBF method seems to only yield more accurate
Table 4: Errors for (22) with IC $u_0(x) = \sin(x)$ and periodic BCs at $t = 10$.

<table>
<thead>
<tr>
<th>kernel</th>
<th>Method</th>
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<td>SAT</td>
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<td>1.5E-2</td>
<td>1.6E-3</td>
<td>1.9E-4</td>
</tr>
</tbody>
</table>

results if quintic kernels are used, but not for cubic kernels. This is related to our previous observation in \S 6.1.2 that the usual RBF method seems to be dissipative for cubic kernels but not for quintic kernels; see figures 2b and 2d. Yet, while the usual RBF method might be slightly more accurate than the SAT-RBF method, its stability cannot be ensured for other kernels. For the SAT-RBF method, on the other hand, stable computations are ensured for all kernels. In fact, in contrast to the SAT-RBF method, we can note that the average order of the usual RBF method decreases when going over to quintic kernels — which actually should increase the order.

Figure 3: Numerical solutions for the acoustic problem (67) with periodic BCs and ICs (69) at $t = 0.5$.
6.3. Extension to Two-Dimensions: Scalar Advection With Zero-Inflow Boundary Conditions

As a last test case, we address the extension of the SAT-RBF method by considering the scalar two-dimensional linear advection problem

$$\partial_t u + \partial_x u = 0,$$

$$u(0, x, y) = \sin(4\pi x) \left(1 - \frac{1}{2} \sin(2\pi y)\right),$$  \hspace{1cm} (70)

$$u(t, 0, y) = 0,$$

on $\Omega = [0, 1]^2 \subset \mathbb{R}^2$. The exact solution is given by

$$u(t, x, y) = \begin{cases} 0 & \text{if } x \leq t, \\ u(0, x-t, y) & \text{otherwise.} \end{cases}$$  \hspace{1cm} (71)

Following the discussion in §5.2, an appropriate discretized SAT term is given by

$$SAT = \frac{1}{2} H^{-1} E_w u.$$  \hspace{1cm} (72)

Here, $H$ denotes the diagonal mass matrix given by

$$H = \text{diag}(h_1, \ldots, h_N), \quad h_n = \int_{\Omega} \psi_n(x) \, dx,$$  \hspace{1cm} (73)

with $\{\psi_n\}_{n=1}^N$ being the nodal basis that spans the approximation space $V_{N,m}$; see §3.2. Furthermore, $u$ is the vector containing the nodal values of $u_N$ at the centers $x_n$, $n = 1, \ldots, N$, and $E_w$ is the matrix defined by

$$E_w(n_1, n_2) = \begin{cases} 1 & \text{if } x_n = (0, y_n)^T, \\ 0 & \text{otherwise.} \end{cases}$$  \hspace{1cm} (74)

Figure 4 illustrates the results of the usual RBF and SAT-method at times $t = 0.2631$ and $t = 0.5263$. In all cases a quintic kernel has been used. We performed the same test cases also for a cubic and a multiquadric kernel. The results were essentially the same and are therefore not illustrated here. Yet, the $L^2$-norms of the error at time $t = 0.52632$ for all three kernels are reported in Table 6.

From these errors, it can be noted that the SAT-RBF method yields more accurate solutions compared to the usual RBF method in all cases. The increase in accuracy by going over to the SAT-RBF method is only small for this problem. Still, this test demonstrates that the RBF-SAT approach can also be used to construct provable stable RBF methods in higher dimensions without decreasing the overall accuracy of the usual RBF method. Finally, it should be noted that the CFL number had to be decreased (from 0.1 for the usual RBF method) to 0.01 for the SAT-RBF method, in order to obtain stable computations. This might be considered as a drawback of the proposed SAT-RBF method. At the same time, a reduced time step size is a well-known byproduct of adding dissipation (which is done by the SAT term). Future works might elaborate on this observation.

<table>
<thead>
<tr>
<th>kernel</th>
<th>Method</th>
<th>$\ell^1$-errors</th>
<th>$\ell^\infty$-errors</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td>$N$</td>
<td>$O$</td>
</tr>
<tr>
<td>cubic</td>
<td>usual</td>
<td>2.3E-1</td>
<td>1.9E-2</td>
</tr>
<tr>
<td>SAT</td>
<td>usual</td>
<td>1.1E-1</td>
<td>1.9E-2</td>
</tr>
</tbody>
</table>

Table 5: Errors for the acoustic problem (67) with periodic BCs and ICs (69) at $t = 0.5$. 

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7. Concluding Thoughts

In this work, we investigated stability properties of RBF methods for linear advection problems. In particular, stable incorporations of BCs were addressed. Classically, in RBF framework, BCs are implemented strongly. This can trigger stability issues in the underlying RBF methods, however. Here, we demonstrated how such issues can be remedied by adapting some well-known techniques from the FD and FR communities. This revealed some, so far, unexplored connections between RBF methods and classical FD and FE schemes. In the process, we proposed two novel RBF approaches, respectively referred to as FR-RBF and SAT-RBF methods. By weakly imposing the BCs through numerical fluxes (in the FR-RBF method) or SATs (in SAT-RBF method), linear stability could be proven for these schemes. A list of numerical simulations support our theoretical findings and demonstrate the advantages of using stable RBF methods compared to usual (unstable) ones. Yet, it should also be stressed that particular methods proposed in this work should not be considered as some kind of ‘ultimate’ RBF schemes for linear advection equations. Rather, we hope that our investigation will pave the way towards a more mature stability theory for RBF methods.

Future investigations might address SBP-like conditions for RBF methods. In particular, a discussion on how the kernel and point distribution could be selected (optimized) to ensure such an SBP property would, in our opinion, be of great interest.
Acknowledgements

Jan Glaubitz was supported by the German Research Foundation (DFG) under grant GL 927/1-1. Philipp Öffner was supported by the UZH Post-doc research grant.

References


