PS #1 — Linear algebra overview and vector norms

Due: 1/16/25, 11:59 PM *Instructor: Jonathan Lindbloom*

Problem 1. *Part (a):* Write a detailed description of the singular value decomposition (SVD) of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$. In your description, be sure to touch on the following points: (1) When does the SVD exist?; (2) Is the SVD unique?; (3) How does the SVD reveal the four fundamental spaces $\mathrm{col}(\mathbf{A})$, $\ker(\mathbf{A}), \mathrm{col}(\mathbf{A}^T), \ker(\mathbf{A}^T)$, and how does it reveal the column rank of $\mathbf{A}$?; and (4) one other property of the SVD that you find interesting.

Part (b): Present (on a chalkboard, or via a notetaking app on Zoom) your description of the SVD to at least one other student in Math 56; list who you presented to and anyone who you listened to. Feel free to revise your description of the SVD with any feedback from your peer(s).

Problem 2. *Part (a):* Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, and let $\mathbf{Q} \in \mathbb{R}^{n\times n}$ be an arbitrary orthogonal matrix. Show that the *similarity transformation* $\mathbf{QAQ}^T$ has the same eigenvalues of $\mathbf{A}$, i.e., the eigenvalues are not disturbed.

Part (b): Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, and let $\mathbf{Q}_L \in \mathbb{R}^{m\times m}$ and $\mathbf{Q}_R \in \mathbb{R}^{n\times n}$ be arbitrary orthogonal matrices. Show that $\mathbf{Q}_L\mathbf{A}\mathbf{Q}_R$ has the same singular values of $\mathbf{A}$, i.e., left or right multiplication by orthogonal matrices does not disturb the singular values.

Problem 3. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$, $\mathbf{x} \in \mathbb{R}^n$. In "Big O" notation, how many flops are required to compute the matrix-vector product $\mathbf{Ax}$ using the standard algorithm? What about the matrix-matrix product $\mathbf{AB}$? Provide reasoning for your answer.

Problem 4. Let $\mathbf{u}, \mathbf{x} \in \mathbb{R}^n$. *Part (a):* What is the rank of $\mathbf{U} = \mathbf{u}\mathbf{u}^T$? What are its eigenvalues?

Part (b): In "Big O" notation, how many flops are required to compute $\mathbf{z} = \mathbf{Ux}$ when computed as $\mathbf{z} = (\mathbf{u}\mathbf{u}^T)\mathbf{x}$? What about when computed as $\mathbf{z} = \mathbf{u}(\mathbf{u}^T\mathbf{z})$?

Problem 5. Let $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n\times k}$ with $\mathrm{rank}(\mathbf{U}) = \mathrm{rank}(\mathbf{V}) = k$. What is $\mathrm{rank}(\mathbf{U}\mathbf{V}^T)$?

Problem 6. Show that $\|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty$ for any $\mathbf{x} \in \mathbb{R}^n$.

Problem 7. Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix and let $\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{C}} := \mathbf{u}^T\mathbf{C}\mathbf{v}$ be the $\mathbf{C}$-weighted inner product for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Verify that $\|\cdot\|_{\mathbf{C}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathbf{C}}}$ satisfies all properties of a norm on $\mathbb{R}^n$.

Problem 8. Let $\mathbf{D} = \text{diag}(d_1, \ldots, d_n)$ where $d_i > 0$ for each i, and let $\|\cdot\|_{\mathbf{D}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathbf{D}}}$. Show that $\|\cdot\|_{\mathbf{D}}$ is equivalent to $\|\cdot\|_2$, i.e., find constants C_1 and C_2 such that

$$C_1 \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_{\mathbf{D}} \leq C_2 \|\mathbf{x}\|_2 \tag{1}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Problem 9. Let $\mathbf{C}$ and $\|\cdot\|_{\mathbf{C}}$ be as in Problem 7. Show that $\|\cdot\|_{\mathbf{C}}$ is equivalent to $\|\cdot\|_2$, i.e., find constants C_1 and C_2 such that

$$C_1 \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_{\mathbf{C}} \leq C_2 \|\mathbf{x}\|_2 \tag{2}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Problem G1. Let $\mathbf{C}$ and $\|\cdot\|_{\mathbf{C}}$ be as in Problem 7, but now let $\mathbf{C}$ be only symmetric semipositive definite with $\text{rank}(\mathbf{C}) = r < n$. Show that $\|\cdot\|_{\mathbf{C}}$ as previously defined is no longer a norm on $\mathbb{R}^n$, but does satisfy the definition of a norm on the subspace $\text{col}(\mathbf{C}) \subset \mathbb{R}^n$.

Problem G2. Prove at least one part of the Courant-Fischer theorem.