

COMPUTING WITH QUADRATIC FORMS—THE PROGRAMS FROM THE ZAK PROJECT

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1. INTRODUCTION

In the 1990s Rudolf Scharlau and I had a joint project concerned with computations for and with integral quadratic (and later also hermitian) forms over \mathbf{Z} and also over the rings of integers of (mainly quadratic) number fields. The project originated in Rudolf Scharlau's group of Diplom and doctoral students, where it also got its name ZAK, I don't really now why (from the German word Zahlkörper (number field) perhaps?). It turned later into a project funded by Deutsche Forschungsgemeinschaft; in this time Alexander Schiemann worked for the project, coordinated the programming work and wrote several C++ programs. The programs developed in the project were the basis of the articles [9, 10, 8, 12]. After Schiemann left academia we continued to use the programs for a while on our HP-UX workstations. When these went out of service it turned out to be difficult to adapt the programs to other environments; I will describe some of the problems later. All experiments I did now are under Linux (Suse 11.0) using gcc 4.3. The present new interest in such computations, in particular in the SAGE project, raises the question whether it is worthwhile (and possible) to revive these programs.

2. THE PROGRAMS

The programs in the ZAK project are centered around the task of classifying positive definite quadratic and hermitian forms. Classifying means here: Enumerate a full set of representatives of isometry classes of forms of fixed dimension, either in a fixed range of discriminants or even in a fixed genus.

2.1. Class-list. The case of ternary quadratic forms is somewhat atypical: Here one can (following Seeber) sharpen the usual reduction conditions on a matrix in such a way, that each isometry class contains a unique matrix satisfying the full set of conditions. Consequently, one can just enumerate the matrices of fixed determinant satisfying all the conditions to obtain the desired list. This is done in the programs in our folder Class-list: We have **ternrep**, which takes as input a positive definite ternary form and outputs the unique reduced form in its class, and **ternclass**, which lists all ternaries in a given range of discriminants (with options to list only even, only primitives etc.) Both programs are in C, were written by Alexander Schiemann and are functional (recompiled under Linux using gcc 4.3 and run). Schiemann developed them during the work on his thesis which was later published as [11] and modified them while working with ZAK. A similar tool is **rep-list**; this generates representatives of genera of forms of specified signature in a given discriminant range. I could compile this program but had runtime errors, probably due to the bigint-routines (see the section about problems).

2.2. Isometry problem. In higher dimensions there is no unique reduction. Moreover, from dimension 5 on the reduction conditions become impractical, from dimension 8 on one doesn't even have an explicit finite set of reduction conditions. One can of course still use the LLL algorithm to keep the coefficients of a representative manageable but this does not even come close to finding unique representatives. It is therefore necessary to test quadratic forms for equivalence efficiently. The programs **isom** and **auto(m)** of Bernd Souvignier (who is now, i. e. in 2009, in Nijmegen), described in [6], have been used in our project, with marginal modifications made by Schiemann. The program **isom** tests for equivalence, the program **auto(m)** determines the group of units (automorphisms) of a given set of forms. The testing of sets of forms can be used to perform equivalence tests over rings of (quadratic) integers. As far as I know these programs are also still part of MAGMA.

2.3. Listing forms—Isolist. During the project a variety of routines was developed for computing various invariants of a set of quadratic forms (also called lattices in the sequel), for example dual lattice, root system, order of the automorphism group, list of short vectors, theta series of degree one and two or Jacobi theta series, dimension of the space of modular forms generated by these theta series, closure of this space under Hecke operators, mass of the set of forms at hand. Schiemann wrote a C++-program **isolist** which takes as input a list of quadratic forms, extracts a set of representatives of the different classes and, depending on options handed to the program, computes various invariants of the forms and organizes the representatives into a list.

More precisely and quoting from Schiemann's documentation, **isolist** offers the following possibilities:

"isolist" is a program to sort \mathbf{Z} -lattices by isometry and/or compute data (theta-series, order of automorphism group) belonging to these lattices. It can also compare the span of (different kinds of) theta-series of a set of lattices with the span of theta-series of a subset.

isolist performs in different stages (some of which are optional):

Stage 1: Parsing of standard input for \mathbf{Z} -lattices until end-of-file or `&End` is read. Result is a list of lattices with some related data that was also read from standard input.

Stage 2: Perform reduction on the lattices, check the list for isometric lattices and delete all but one representative of isometric ones. Depending on the options some invariants of each lattice may be computed to simplify tests for isometry. Result is a list of representatives of the classes of lattices that were read.

Stage 3: Compute the theta-series of the representatives (if not known already) and sort the list depending on the theta-series. Compute additional information about the spaces of different kinds of theta-series belonging to the lattices and print this information on standard output.

Stage 4: Compute additional invariants of each representative and print the representative together with the invariants. The order in which the lattices are processed and printed depends on their theta-series or on their number.

The output is in a format that can be processed by **isolist** again. So you may accumulate information in different runs of **isolist** (without loss of performance, since the tags `&begin_block/&end_block` are used to mark the list sorted by isometry).

To give an impression of the way in which Schiemann organized his programs, here is a file list of the folder Isolist:

```
assign.h
asub.c
asub.h
asub_isolist_templ_I.cc
asub_templ.cc
asub_templ.h
bigbin_modular_form.cc
bigbin_modular_form.h
bigfloat_adapt.h
bigfract_call.cc
bigfract.cc
bigfract.h
bigint_adapt.h
bin_IO.h
bin_modular_form.cc
bin_modular_form.h
bin_theta_series.cc
bin_theta_series.h
blocked_vectorC_stack_templ.cc
blocked_vectorC_stack_templ.h
blocked_vectorV_stack_templ.cc
blocked_vectorV_stack_templ.h
block_link_templ.cc
block_link_templ.h
classlist_util.cc
classlist_util.h
COPYING-2.0
COPYRIGHT_isolist
cp_rm_diff
datatypes_templ_I.cc
datavec.cc
datavec_templ.cc
datavec_templ.h
fract.cc
fract.doc
fract.h
Global_DimensionC.cc
Global_DimensionC.doc
Global_DimensionC.h
Glob_int2.cc
handle_uninit_templ.cc
handle_uninit_templ.h
include_param_isolist.h
integer_util_templ.cc
integer_util_templ.h
int_PF.cc
int_PF.h
iso_aut_source.c
iso_aut_source.h
isolist.cc
```

isolist.doc
isolist.h
isom-list.cc
isom-list.doc
isubspace_templ.cc
isubspace_templ.h
isubspace_templ_I.cc
jac_theta_series.cc
jac_theta_series.h
lidia_param.h
list_errors.cc
list_errors.h
list_with_positions_templ.cc
list_with_positions_templ.h
lower_triangleC_templ.cc
lower_triangleC_templ.h
Makefile
maklib.h
malloc_redefine
malloc_unredefine
math_types.h
matrixC.cc
matrixC_templ.cc
matrixC_templ.h
matrixC_templ_I.cc
matrix_row_ref_templ.h
matrix*_templ.doc
matrix_templ_I.cc
matrix_util_templ.cc
matrix_util_templ.h
matrixV.cc
matrixV_templ.cc
matrixV_templ.h
new_nothrow.cc
new_nothrow.h
new_version_file
option_util.cc
option_util.h
option_util_isolist.cc
option_util_isolist.h
orbits_iter_p.cc
orbits_iter_p_isolist_templ_I.cc
orbits_iter_p_templ.cc
orbits_iter_p_templ.doc
orbits_iter_p_templ.h
ppointer_templ.cc
ppointer_templ.h
README.1st
README.AUTO
README.ISOM
redu_util_isolist_templ_I.cc
redu_util_templ.cc

```

redu_util_templ.h
s_malloc.c
s_malloc.h
s_malloc.usage
strongly_mod.c
strongly_mod.h
sym_matrixC_templ.cc
sym_matrixC_templ.h
sym_matrix_I0.cc
sym_matrix_I0.h
Tests
theta_series.cc
theta_series.h
types.h
unmaklib.h
vectorC.cc
vectorCp_templ.h
vectorC_templ.cc
vectorC_templ.h
vector_templ_I.cc
vectorV.cc
vectorV_templ.cc
vectorV_templ.h
VERSION
version_isolist.h
Zclassdata2.cc
Zclassdata2.h
Z_isoautC_templ.cc
Z_isoautC_templ.h
Z_isoautC_templ_I.cc
Z_isoaut_options.h
Z_latticeC_class_templ.cc
Z_latticeC_class_templ.h
Z_latticeC_class_templ_I.cc
Z_latticeC_shortvecs_templ.cc
Z_latticeC_shortvecs_templ.h
Z_shortvecs_orbit_util_templ.cc
Z_shortvecs_orbit_util_templ.h
Z_shortvecs_orbit_util_templ_I.cc

```

Attempts to recompile Isolist failed, see section Problems. A different program for organizing lists of lattices is **invar**, it is essentially a part of Hemkemeier's routine **tn** and will be discussed together with it in the next subsection.

2.4. Neighbouring lattices. The method of neighbouring lattices has been introduced by Kneser in [4]. Its use for computer calculations was described in [13, 8, 12]. Over \mathbf{Z} it can be used to determine all classes in a given genus, over the integers of number fields one has to be careful about obtaining all spinor genera in the genus. Our project produced the routines **tn** calculating neighbours at the prime 2 of a \mathbf{Z} -lattice and thereby obtaining all classes in the genus, **pn**, **pn-sh** for doing the same using neighbours at some prime p , **qn** for doing the same over a real quadratic field, and **hn** for doing the same for hermitian forms over the integers of an imaginary quadratic field.

Of these, **hn** could not be recompiled (see section Problems). Some years ago Abshoff created an executable which was used on Linux PCs in Dortmund and Saarbrücken; I haven't tested recently whether it can still be used.

tn, written by Boris Hemkemeier, has been maintained by him and is available in the web [2]. The programs **pn**, **pn-sh**, **qn** could be recompiled, running has not yet been tested, except for one or two runs with **pn-sh**. The program **tn** has more utilities than **pn**, **qn** for generating additional information, e.g. root systems of lattices. These facilities of **tn** have been collected by Schiemann in the program **invar** which takes as input a list of lattices generated by one of the other programs. **invar** could (with marginal changes) be recompiled but has not yet been tested for running.

I include the doc-file of **hn** to give an impression of what it does when properly activated:

```
/*
  Copyright (C) 1997  Alexander Schiemann
  (try 'hn --copyright' for more information)
*/

this file describes VERSION 2.8.6 of hn (15.3.99)

SHORT INFO:
=====

This program (hn) deals with positive definite integral hermitian
forms over an imaginary quadratic field.
It collects all functionality that I have implemented for
hermitian forms. This is:
- filter for different IO-formats (see: --invar, -I, -O )
- reduction of hermitian forms with respect to  $Gl(n, O_K)$  (see: --invar,
  --reduce, --herm_lll )
- calculating invariants for hermitian forms, e.g. automorphism group,
  theta-series: (see: --invar, --autom, --groups, --shell -O )
- sorting of lists of hermitian forms for isometry
  (see: --invar, --isom_test )
- generating the neighbourhood (representatives of all classes)
  of a given form
  ('unmodified normal mode' of the program).
- heuristic search for forms with large minimum in a neighbourhood
  (modifications of the 'normal mode').
(see also below under 'OPTIONS AFFECTING THE TYPE OF ACTION ...')
```

In its normal mode this program computes one or more special genera of integral hermitian forms over an imaginary quadratic field K using the neighbour method of Kneser. This method for the hermitian case is described in detail in the paper "Classification of hermitian forms with the neighbour method" that can be obtained at <http://www.math.uni-sb.de/~aschiem>. In its normal mode (with isometry tests and construction of orbits of neighbour vectors under the action of the automorphism group) the program works in dimensions (over K) 2, ..., 10 (,11,12) with neighbours at small primes P . The runtime is $O(N \cdot M \cdot I)$, where N is the number of neighbours at P of a lattice, M the number of classes in the union of

special genera that will be computed and I the time consumed by one isometry test or construction of an automorphism group. Isometry tests and construction of automorphism groups are subroutines based on the programs 'isom' and 'bravais' (alias 'autom') by BERND SOUVIGNIER 1995, whom I want to thank for the permission to use the code of his programs.

SOME REMARKS:

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There are many options and not all combinations are meaningful. Some options cancel other options without warning. At the beginning you should rely on the defaults and only specify options for the 'overall choices'. Use --defaults to find out which options really are in effect.

If you use 'hn --invar' you should read the help given with 'hn --invar --help' to understand how to pass a list of lattices (possibly with known invariants) to 'hn'. Usually you will write the specification of the imaginary quadratic field, the dimension of the hermitian space and one or more lattices into a file and say 'hn --invar < file'.

Critical applications may need fine-tuning.

- pass known invariants of lattices to 'hn --invar' as described in 'hn --invar --help'. Especially known automorphisms may speed up the calculation of the automorphism group and isometry tests (use the 'Aut_subgroup' invariant)
- try to collect information in successive runs, starting with reduction only. optimize the parameters of each subalgorithm.

DETAILED DESCRIPTION OF OPTIONS:

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Feel free to try combinations of options. Inconsistent combinations should result in warnings or errors (but may sometimes also lead to strange results).

We made quite an effort to ensure that the output is not misleading whatever strange input was made.

Use --defaults to learn which of the options actually remain in effect.

INVOCATION ([...] stands for optional):

hn [options]

INPUT depends on options. In the normal mode, there will be prompts for input on standard output. The input is

D dim lattice prime

where:

- D : specifies an imaginary quadratic field $K=\mathbb{Q}(\sqrt{D})$
- dim : dimension (over K) of the hermitian space
- lattice : [list of ideals] [{factor}*] gram_matrix
for an exact description of the input format start the program with the --verbose (-v) option

prime : prime_number or ~prime_number (for the conjugate prime ideal)

OPTIONS : [[-]single_character_options] [[-]key_character[inner_options]]
 [--word_options]

single-character options: [hH?svltn A'n' T'n' S'n' G'n' D'n']

key-character options : [O[csbwpPLTOU] I[bwPLB]]

word options : [--invar --recover --recover'filename'
 --neighbours'n' --neighbour_orbits'n'
 --only_even --only_odd
 --max_stable'n'--isom_test --no_isom_test --help
 --Help --silent --verbose --neighbour_verbose
 --neighbour_silent --graph_max'i'
 --trace_isom_search --no_trace_list
 --blank_format --w_format --pari_format
 --shells'i' --Depth'i' --mass --no_mass ...]

Options can be given as single character options (with an optional leading '-' and optional space between) or as a keycharacter followed by a keyword or an inner list of character options (ending with a blank) or as a word-option (that stands for a combination of other options in some cases).

Single character options and word-options may have arguments.

Most options have inverse options. In case of competing options, the last given overrides the others (important for --recover (see below)).

OPTIONS AFFECTING THE TYPE OF ACTION THAT 'hn' WILL TAKE:

 (if none of these options is given, the expansion of the (whole) neighbour-graph takes place, this is the normal action of 'hn'. Modifications of the normal mode are described below.)

--version : show the version number and date

--copyright : inform about copyright

[-]h : give this help

[-]H

[-]?

--help

--Help

--help --invar : give help about the input parsing with --invar

--help --isom : prints original description of the isometry test for sets of bilinear forms over Z (by Bernd Souvignier). This test is used within the isometry test of the lattices over the ring of integers of a quadratic field here. You may set the option for this algorithm using 'set_isoaut_options(...)' (see below).

--help --autom : prints original description of the computation of the automorphism group of a set of bilinear forms over Z (by Bernd Souvignier). This test is used within such

computations for lattices over the ring of integers of a quadratic field here.

You may set the option for this algorithm using
'set_isoaut_options(...)' (see below).

```
--defaults          : show the default settings, i.e. show the settings
                      that would take place with the given options except 'defaults'
                      (the defaults may depend on given options!)
```

```
--recover'filename' : (no space between recover and 'filename')
                      (not interactive) if possible a run of hn generates a logfile
                      to save partial results in case of a crash. A new logfile gets
                      the first non-existing name from hn.log,hn.log1,hn.log2..
                      If a run ends successfully the log file is destroyed.
                      With the recover option a crashed run that produced the logfile
                      'filename' is resumed and new results are appended to that file.
                      A recover-job will use the same options as the original
                      job with the new options appended to the old ones and
                      possibly overriding them. All meaningful changes of parameters
                      are allowed with --recover and all other combinations of
                      options (e.g. --only_odd recovered with --only_even) will
                      (hopefully) lead to an error message and abort the program.
                      The representatives of the classes and their order in the
                      output of a recovered job may be different to that of a
                      completed job with the same input. This is because a
                      recovered job may recalculate the last (incomplete)
                      expansion with another representative, if a better
                      representative was found during the last expansion and
                      no '&Neighbour #n'-directive with n>1 has been saved in
                      the logfile. Also the counter of the max_stable option is
                      reset when a job is restarted.
                      The logfile is removed after a complete run unless
                      the expansion was terminated because of --stop_mass.
```

```
--recover          : (not interactive) same as above but the filename
                      will be the first of hn.log,hn.log1,...,hn.log19 with an
                      existing file.
```

```
--invar            : (not interactive) do not construct lattices but read
                      lattices (and related data) from standard input, perform reduction/
                      tests for isometry and/or compute invariants of these lattices
                      (depending on other options).
                      The input is parsed according to the rules described with
                      'hn --help --invar', everything but lines following special 'tags'
                      is ignored. Output and log-files are in a suitable format to
                      be used as input to 'hn --invar'.
                      If a logfile is passed to hn --invar, the invariants
                      of the lattices found up to the crash are computed.
```

```
--neighbours'n'    : (interactive) Compute at most 'n' neighbours of
                      the start lattice without orbit construction and without tests
                      for isometric lattices.
```

--neighbour_orbits'n' : (interactive) As above but compute the automorphism group of the start lattice and compute only neighbours belonging to different orbits under the action of the automorphism group.

MODIFICATIONS OF THE NORMAL MODE (all interactive):

 --min_search['L' ['M' 'S']] : heuristic option to search for lattices with large minimum in large genera (which can not be classified completely). Implies --maxmin -G0 and --no_intermediate_mass. 'L' and 'M','S' are positive integers.

A run with --min_search writes not only a logfile (see --recover) but also a file named "largest_minimum.D-'disc'.dim'DIM'.det'DET'.tmp" (with the values 'disc' the discriminant of K, 'DIM' the K-dimension and 'DET' the discriminant of the hermitian lattices. Whenever a lattice with a larger minimum is found, the lattice is appended to this file (thus a file of such a name may be altered by hn). This file is the usual output. If a run ends, it gives the classes that were expanded on standard output. In general, this will not be the whole neighbourhood.

The expansion strategy is plainly as discribed below with --maxmin, but the list for the unexpanded classes has limited length 'L'. If it grows longer, the last class in this list (i.e. that with the most short vectors up to the current bound 'maxmin') will be dropped.

Let at a moment 'l' be the length of the list for the unexpanded classes. Given a new neighbour with minimum 'm', this neighbour will be dropped (without any notice) if (M,S) is given and $m < M$ and there are $\geq S$ unexpanded lattices better than the new one.

Otherwise

if $l < L$ or ($l = L$ and the new lattice would NOT be last of (unexpanded without the one being expanded) in the order described with --maxmin)

the new lattice is checked against all classes for isometries and is added to the list of all and of the unexpanded classes if it belongs to a new class.

if ($l_{\text{new}} = l + 1$) == L, then the last item (without the one being expanded) is dropped from unexpanded.

otherwise (i.e. $l = L$ and the new lattice has the 'most short vectors') the lattice is dropped (without any notice).

NOTE: the list of expanded classes is not limited and

isometry tests take place. The logfile may contain lattices that are dropped later on. It may be recovered (see above). You may change the --min_search parameters. This will NOT prevent lattices marked as &dropped in the logfile from being dropped, even if the new queue of unexpanded lattices is longer.

If the isometry tests fail or need to much time, you may combine this option with the following ones, which imply --min_search:

--no_isom['k'] : never test for isometry.

Every new lattice is dropped under the conditions

described above. If not (i.e. if the algorithm stated above would

start isometry tests), the new lattice is dropped
 IF the number of expanded classes with the same theta series
 (up to a bound 'B')+ the number of such lattices that had ever been
 added to the queue 'unexpanded' is $\geq k$ (this is to avoid infinite
 loops). Otherwise it is appended to unexpanded.
 The bound 'B' used above is the current value of theta_bound (see
 also --maxmin). Whenever 'B' changes, all theta-series-counters are
 re-evaluated. You should use -T to give a reasonable start value
 for B.
 It does not seem very useful to restart such a run, although
 this should be possible (in that case all the theta-series-counters
 are set to the number of matching thetaseries in unexpanded).
 --jump['k'] : 'k' is an optional integer. Without this option, to
 every class being expanded all neighbours will be computed. If the
 number of neighbours is very large, that may be unefficient. If a
 better lattice shows up, the current expansion is aborted after 'k'
 more neighbours have been constructed unless another even better
 lattice is found. Whenever this occurs, the counter for 'k' is reset.
 If --no_isom is given, the theta-series-counter of the theta series
 belonging to the currently expanded lattice is increased when the
 expansion is aborted because of --jump exactly as in the case of a
 completed expansion. On the contrary, if --no_isom is not active,
 the currently expanded lattice will not be marked 'expanded'; it
 may be expanded again later, if all better lattices have been
 expanded.
 To switch off --jump, give a negative value for 'k'.

NOTE: successive options --min_search --no_isom --jump will override
 the previous ones, but only with those values that are explicitly
 given. The first option --min_search will provide default values if
 all or some of the parameters are omitted. Repeated calls will only
 override parameters with new values, if those are explicitly given.
 e.g. --min_search201 5 17 --min_search300 is --min_search300 5 17

--exclude_by_shortvecs'b' 'm' 'f': implies --min_search: 'b' and 'm'
 are non-negative integers and 'f' a floating value between 0 and 1.
 Let L denote the currently expanded lattice (the father) and S be
 the set of vectors of L with length $\leq b$.
 If $\#S \leq m$ then a neighbour L(x) of L is ignored if S intersected
 with L_x:=(L intersected with L(x)) has more than $\#S * f$ elements:
 In short: We may exclude neighbours only by computing how many of
 the vectors in S are in L_x (which is only the computation of one
 scalar-product for each of the vectors).
 If 'b' is 0 then the option is reset.

--pre_exclude_bound'b' 'm': implies --min_search: 'b' and 'm'
 are non-negative integers. After a neighbour passed the 'exclude_
 by_shortvecs' test (described above) the theta-series is computed
 up to length $\leq b$. If there are more than 'm' vectors up to length
 'b' then the lattice is dropped.
 If 'b' is 0 then the option is reset.

OTHER GENERAL OPTIONS:

[-]s
 --silent : do not print input requests and explanations
 [-]v
 --verbose : print input requests and some explanations on cerr

 --reduce : reduce the lattices that are read from a logfile or
 from standard input (the start lattice/all lattices with --invar)
 --no_reduce : do not reduce the lattices given.

 --no_autom : never compute automorphism groups. This option implies
 -A INT_MAX --no_mass and disables the orbit construction on the set
 of neighbour-vectors.
 --autom : allow construction of automorphism groups

 --isom_test : only with --invar: test the given classes for isometric
 ones and give a list of the 'best' representatives
 --no_isom_test : only with --invar: do not test the given classes (from
 standard input) for isometric ones

 --check_disc : only with --recover: check the discriminant of lattices
 read from the logfile
 --not_check_disc : only with --recover: do not check the discriminants

 --groups : whenever an automorphism group is computed it is stored.
 This data is useful to avoid recomputation of the automorphism group
 as otherwise necessary for orbit computations. It may also be used
 to speed up tests for isometry (see --iso_use_known_auts). This option
 will increase the storage consumption if there are many classes.
 NOTE: The order of the group of automorphisms will be stored whenever
 it becomes known, regardless of --groups.
 NOTE: the information about the group is lost, if the representative
 is changed because of reduction or by &replacing...
 This option also enables output of group generators to a logfile,
 for a description of this output see -OG below.
 Reading of automorphisms from a logfile or from standard input is
 controlled by -IG (see below). Printing to standard output is
 controlled by -OG (see below).
 --no_groups : no group generators are stored nor printed in the logfile

OPTIONS AFFECTING THE INTERPRETATION OF INPUT:

[-]I[bwPLB] : affects the input of integers from O_K and the format
 of the hermitian matrices read from stdin (not from a logfile
 when read with --recover and not with --invar):
 input options specified by (one or more) characters appended to 'I':
 b : input numbers in O_K in blank_format (e.g. 1 0, 0 1, 3 -2)
 w : input of numbers in O_K and K in w_format (compatible to PARI)
 (e.g. 1, w, 3-2w, 2-5w, 2-3*w or 1/3-2w, 1/2+3/5*w)
 P : Gram-matrix input is in PARI mode (e.g. [*,* ; *,*])
 L : Gram-matrix input is the lower triangle of the hermitian matrix)

B : " " " is given as a full matrix or a lower triangle. The end of input for a row is given by newline
 G : store elements of the automorphism group that are read from a logfile or with --invar from standard input. For the format see 'hn --invar --help' (either the generators of the full group of automorphisms or just some automorphisms may be specified).
 WARNING: of course, the generators are no invariants of the class. Nevertheless, they may be given after &invariants in the logfile. Then they belong to the most recently given Gram matrix of this class. The automorphisms are given as matrices with respect to the same base as the Gram matrix. The automorphism matrices G_i are subject to $G_i * H * (\sim G_i).transpose() == H$ (where H is the Gram matrix).
 NOTE: the information about the group is lost, if the representative is changed because of reduction or by &replacing...
 N : ignore automorphisms given in a logfile or in standard input

--blank_format : I/O of numbers in O_K as two components separated by Gram matrices as lower triangle

--w_format : I/O of numbers in O_K and K using 'w' (e.g. 1, w, 3-2w, Gram matrices as lower triangle)

--pari_format : I/O of numbers in O_K and Gram matrices in PARI-format, (e.g. [2,3-2*w ; 3+2*w, 4] (given conjugate(w)=-w))

--multiply_by'I' : multiply any given lattice (i.e. multiply every ideal of a given lattice) with the ideal 'I', where 'I' is an ideal given in standard format (Note: 'I' will not be processed until the quadratic field K is initialized. at startup everything enclosed by <...> (including < and >) is assigned to a string that later defines the ideal). Note: in most shells it will be necessary to enclose the arguments by "\"..\"", because '<' and '>' stand for the redirection of input. To learn about input of an ideal, start the program with --verbose. Multiplying will affect every form read from standard input (e.g. with --invar) but has no effect when logfiles are read

--scale_by'f' : scale any given hermitian form (i.e. multiply the Gram matrix) by the factor 'f', where 'f' is a fract written as numerator/denominator without blanks. This will change the hermitian space where the form lives in iff f^{\dim} is not in $\text{Norm}(K)$. Scaling will affect every form read from standard input (e.g. with --invar) but has no effect when logfiles are read. The result of multiplying and scaling (if both --multiply_by and --scale_by are given) must be integral for every lattice that is entered.

OPTIONS CONTROLLING THE NEIGHBOUR CONSTRUCTION:

```

--neighbour_verbose : print each neighbour and the vector used to construct
                      it to the logfile or to standard output in case of
                      --neighbours or --neighbour_orbits
--neighbour_silent  : do not print the neighbours

--only_even         : only if neighbours are computed at a prime above 2
                      and the start lattice is even: Only even neighbours are
                      generated. This will result in computing all even
                      lattices in the neighbourhood, because the even lattices
                      are a connected subset in the neighbour graph. This option may
                      not be combined with --recover (only_odd or only_even of the
                      original run will be remembered automatically)

--only_odd          : only if neighbours are computed at a prime above 2
                      and the start lattice is odd: Only odd neighbours are
                      generated. WARNING: The odd lattices are not necessarily
                      a connected subset in the neighbour graph. Possibly not
                      all odd lattices in the neighbourhood are generated
                      This option may not be combined with --recover (only_odd or
                      only_even of the original run will be remembered automatically)

--max_stable'n'     : after 'n' successive expansions of classes in one
                      depth that did not produce a new class, the next class
                      to expand is choosen from another depth (if possible)
                      The 1st depth bigger than the current with an unexpanded
                      class is choosen or (if there is no such class) the
                      smallest depth with an unexpanded class. Here the 'depth'
                      of a lattice means the minimal number of neighbour steps
                      from the start lattice to this lattice along a path that
                      'the program knows about' (not exactly the depth in the
                      neighbour-graph)

--stop_mass'bf'     : expansion of the neighbour graph is stopped if the
                      mass becomes >='bf', where 'bf' is a bigfract (written in the
                      form numerator/denominator without blanks). This option
                      implies --graph_max0 and --intermediate_mass. After the run
                      the logfile is not deleted. If the mass 'bf' was reached and
                      the expansion left incomplete, this is notified in the output.
                      if 'bf'==0, a former stop_mass is erased (can be used to
                      reset this option in a recovered run).
                      if --invar, --neighbours or --neighbour_orbits is given,
                      --stop_mass has no effect.

[-]A'n'
--autom_bound'n'    : the order of the automorphism group is used as an
                      invariant to avoid isometry tests if more than 'n'
                      classes are already found.

[-]T'n'
--theta_bound'n'    : the theta series of each lattice is computed up to
                      norm 'n' to avoid isometry tests. This is just a start value.
                      When 1000 classes have been found, 'n' will automatically be set

```

(increased) such that it is maximal with $\leq 5\%$ of the thetaseries being shorter than 'n' (at that time). In a resumed run this setting will take place at the beginning if there are more than 1000 classes in the logfile. If --maxmin is given, theta_bound may be increased whenever a lattice with larger minimum is found.

--compute_missing_groups: only with --groups and --iso_use_known_auts, redundant if -A0.
the automorphism group of every new class is computed and stored so that it can be used in future isometry tests. NOTE: Differently to -A0 the automorphism groups of new neighbours are not (necessarily) computed, only after isometry testing has been performed and the neighbour found to be in a new class.
This option guarantees that in each isometry test at least the automorphism group of one lattice is known.

--not_compute_missing_groups: switch off --compute_missing_groups

--intermediate_mass : whenever a class is found, the sum of the inverse orders of the automorphism groups is written into the logfile. (implies --mass).

--no_intermediate_mass: the mass is computed only at the end of the run (or never), to avoid intermediate automorphism calculations

--mass : the mass of the family of lattices (i.e. is the sum of the inverse orders of the automorphism groups) is given in output at the end of the run.

--no_mass : the mass is not computed at all (implies --no_intermediate_mass).

[-]G'i'
--graph_max'i' : 'i' is an integer. The adjacence-matrix will be computed up to a class number $\leq i$ (to avoid quadratic storage consumption)

--maxmin['n']['b']] : 'n','b' optional integers ('n' will be set to the maximal minimum+1 of all forms known at the beginning or resuming of the neighbour-construction if it is omitted).
Changes the expansion strategy:
Briefly: Searches for forms with large minima, starts with a strategy to find minimum 'n'. Whenever such a form is found, 'n' is increased by 1 until the bound 'b' is reached. Then more forms with minimum 'b' are searched.
Case $n > 0$: The next class to be expanded is the best (i.e. smallest) with respect to the following total order:
0. a form with $2 * \text{minimum} \geq n$ is better (i.e. $<$) than one with $2 * \text{minimum} < n$.
1. smallest $\#\{x \mid h(x,x) < n\}$ among these with
2. largest minimum and among these
3. minimal number (as detected in the expansion)
Whenever the new largest minimum l becomes $> n$,
 n is set to l or to $\min(b, l)$ if b was given
and the unexpanded classes are reordered.

Whenever 'n' is increased, theta_bound is set to
 max(n-1,theta_bound)
 Case n<=0: Reset this option and resume the usual expansion strategy
 depending on the number and switching depth according to
 --max_stable. Used to reset the option in recovered runs.

OPTIONS CONTROLLING (PARAMETERS OF) SUBALGORITHMS:

```
-----
```

--set_isoaut_options('string'): (FOR EXPERTS, rely on defaults or try
 more than once!) 'string' must not contain ')'.
 and the closing brace must be separated from the next option by
 whitespace. This option can be used to set all parameters of the
 subalgorithms developed from ISOM and AUTOM of Bernd Souvignier
 which compute the automorphism group/perform isometry test for
 sets of bilinear forms over \mathbb{Z} .
 Some options also may be set directly (see below).
 'string' may contain options as described in 'README.ISOM' and
 'README.AUTOM' which you may print using --help --isom or
 --help --autom (see above). Different options in this string
 shall be separated by whitespace. If a suboption of this whole
 option is omitted then reasonable default parameters are used.
 This option is accumulative even interweaved with -D or -B.
 Note: The options will affect all computations of automorphism
 groups as well as isometry tests. Some options are not meaningful
 and will be ignored or overridden depending on the situation.
 more precisely: the -S'n' option with 'n'!=0 leads to the
 computation of a stabilizer subgroup of all automorphisms in
 'autom2'. But such a setting is overridden in all methods of
 herm_lattice_class and hneighbour_orbits. Thus nothing but
 the computation of the automorphism group for output with -OG
 is affected by -S.

[-]D'i'
 --isoaut_depth'i' : 'i' is an integer. the C-subroutines isom2 and bravais2
 (isometries, automorphism groups) work with 'depth i'
 use i=0..6. higher depth increases performance and
 storage consumption of these subroutines

[-]B'i'
 --isoaut_bachdepth'i': 'i' an integer. the C-subroutines isom2 and bravais2
 (isometries, automorphism groups) work with 'bachdepth i'
 use i=0..dim. higher depth increases performance if the automorphism
 groups of the lattices are small

--iso_use_known_auts: changes strategy for tests for isometry: if
 (some) automorphisms of the lattice which is passed to isom2
 as 2nd one are known, these are used to construct orbits of
 the short vectors. This may increase computation time in easy
 cases but should improve performance for most difficult
 cases. implies --groups -IG
 (There is a related parameter 'noaut_norm_threshold' that affects
 cases where iso_use_known_auts is set and only one lattice has known

automorphisms and this lattice has also the smaller maximal diagonal entry in its Gram matrix with respect to a Z-base.
Then minimizing the
bound for the needed short vectors contradicts use of the known automorphisms. This is resolved as follows: If the necessary bound for the short vectors increases by a factor \leq noaut_norm_threshold then this is acceptable and the automorphisms are used (that is: the lattice with automorphisms is passed as 2nd lattice parameter to the Z-isometry test).
by now, noaut_norm_threshold is set to 1.3)

--iso_ignore_known_auts: switch off --iso_use_known_auts (does NOT reset --groups) see also --compute_missing_groups above
/* related hidden parameter (for developers only):
noaut_norm_threshold */

--use_K_linearity : the calculation of the automorphism group of hermitian lattices and isometry test make use of the hermitian structure but thus cannot use reduction of the Z-basis by the usual LLL. Only O_K -linear transformations are done to find a better basis (the hermitian LLL is performed). Switch it off, if the basis found by this means is too bad and causes isom/autom troubles

--Z_or_K_linearity : try Z-reduction of the trace form and use either the hermitian structure or the Z-reduction in isom2/autom2 depending on what strategy leads to smaller lists of 'short vectors' involved in isom2/autom2).

--herm_ulll'i' 'f' : perform the reduction of the O_K lattices by a hermitian version of LLL with blocksize $2 \leq i \leq \dim$ and an LLL-constant $0 < f \leq 1$. (disabled to avoid overflow, if hn is compiled without dynamic integers (LEDA or LiDIA))

[-]S'n'

--storage_bound'n' : the maximal size of the workspace for the construction of the orbits will be 'n' kilobytes. This should fit into the memory of your machine. If the workspace cannot hold all neighbour vectors, the construction of the orbits needs more time.

--orbit_searchdepth'n' : if there are so many neighbours, that the neighbour vectors do not fit into the workspace (see -S), then vectors that have a representative in their orbit that can be found with $\leq n$ iterations of 'representative-search' (see description of orbits_iter_p_templ.h) are no longer held in the workspace. Increasing 'n' increases computation time but allows more vectors to be sorted into orbits (given a fixed -S). If no more space in the workspace is left and the maximal search-depth 'n' is reached, the orbit construction switches to 'pseudo orbits', i.e. simply gives a set of vectors that covers all orbits (as small as is possible with the given limits) but without exactly computing the orbits. If this happens, the order of the orbits cannot be tracked and the adjacency-matrix will not be computed.

running process and look at the backtrace. (If this does not work, restart with the program `hn.debug` (if it exists) and retry).
 -- just try all possibilities listed below

The following functions are known to cause problems without notifying it:
`autom`, `isom2` : although in most cases the runtime depends only on the dimension (over K) of the lattices (≤ 10 , 11 is possible) , there are very rare examples where they fail in dimension 6 (or bigger).
 Try to change depth parameter (`-D'n'`) or Bacher parameter (`-B'n'`).
 For `autom`: switch off the computation in the special situation with `--no_mass`, `-A100000` or in the whole run with `--no_autom`. It may also be useful to improve the reduction using e.g. `--herm_l1l3 0.99` (see below).

`bigherm_lattice::block_l1l1`, `block_l1l2` : In rare cases there is a numerical explosion causing the routine to run for hours or forever. If the action is `--invar` or reading a logfile try to avoid reduction by `--no_reduce`. Or change the parameters of this process, e.g. `--herm_l1l2 0.5`

`herm_lattice_class::vector_bound`, `theta_series::theta_series` :
 computation of the thetaseries up to a bound. Runtime depends on the dimension and the bound. With `--shells` (or perhaps as a default) this bound depends on the current gram matrix, what will cause problems in dimensions ≥ 8 in some cases. Use `--shells'n'` with negative 'n', e.g. `--shells-1`

2.5. Miscellaneous. In this subsection I list (alphabetically) miscellaneous routines which have been programmed in the project.

decomp: The program **decomp**, written by Frank Vallentin, computes the decomposition of a given lattice into its irreducible components. I could recompile it, but the attempt to run it crashed with

```
*** glibc detected *** decomp: double free or
corruption (fasttop): 0x08055458 ***
```

I refrained from attempts to debug it.

gsymbol: The program **gsymbol**, written by Schiemann computes the genus symbol (as defined in [1]) of a given (definite or indefinite) integral quadratic form over \mathbf{Z} . It could be recompiled and ran in simple tests.

herm_mass: The program **herm_mass**, written by Schiemann, computes the mass of the hermitian genus of the sum of m squares over a given imaginary quadratic number field. It could not be recompiled.

perfrk: the program **perfrk**, written by Axel Pawellek, computes the prefection rank (see [5]) of a lattice. I could not compile it at first try, but the only problem seems to be a proper link to **Lapack**, being in a hurry I reserved that for later.

rootssystem: The program **rootssystem**, written by Axel Pawellek, computes the root system of a given lattice. I could not recompile it, due to LiDIA problems.

shvec: The program **shvec**, written by Frank Vallentin, computes lists of short vectors and the theta series of degree 2. According to Vallentin it has been “almost not” tested. I could recompile it but didn’t test it.

sort-list: This program isn't really well documented. It is written by Schiemann and may have been abandoned for some reason (superseded by `gsymbol`?). According to comment lines found in the main program it takes a symmetric matrix S (Gram matrix) from standard input and gives the determinant $\det(S)$, for all primes p dividing $2 * \det(S)$ the p -adic symbol of S and the signature of S . The symbols are those in Conway-Sloane. The 2-adic symbol is the unique symbol given there. I didn't seriously try to compile it.

strongly_modular: This program isn't really well documented. It is apparently written by Vallentin and seems just to test whether the input lattice is strongly modular in the sense of Quebbemann [7]. I could compile but didn't yet test it.

3. PROBLEMS

One rather easy type of problem is the change of environment. Schiemann wrote his programs for our HP-UX workstations with some alternatives of using `egcs` on LINUX machines. Consequently, in the makefiles one has to change all path declarations and some of the flags set. In some cases I experimented with that and was successful. Generally this is something at which I am not particularly good but it should not be a serious problem for an expert.

Apart from that there are two main types of problems

Changes in the C standard: Some changes in the standards for C and C++ require adaptations. Relatively harmless are changes in the names of files that need to be included, e. g. replacing `iostream.h` by `iostream`, including `stdlib.h` and `string.h` in several places, adding `using namespace std;` in several files. In `gsymbol` some other conventions changed, I list them in my comments file in that directory.

More serious are changes in the allowed programming style in C++. In particular, the way in which Schiemann made friend declarations in his templates is now illegal. It seems that this can be fixed by using forward declarations. At least in `gsymbol` that seemed to work:

I changed the original header file `sym_matrix_templ.h` in the following way:

```
/* The following forward declarations inserted by RSP, Jan 2009,
in order to avoid errors of illegal use of friends*/
template<class T>
class sym_matrix;
template<class T>
void ijvertausch(sym_matrix<T>& A,int ii,int jj);
template<class T>
void ijdiag0add(sym_matrix<T>& A,register int ii,register int jj);
template<class T>
int compare(const sym_matrix<T>& A,const sym_matrix<T>& B);
template<class T>
int transform(sym_matrix<T>& A);
template<class T>
void make_primitive(sym_matrix<T>& A,T &factor);
template<class T>
int jacobistep(sym_matrix<T>& A,T& z,T& n,T*& d);
template<class T>
void jacobidet(sym_matrix<T>& f,T& det,int* dimplus,
               int* dimnull,int preserve_matrix);
```

```

/*end inserted forward declarations*/
template<class T>
class sym_matrix{
private:
    // the order of elements is essential for constructor-initialisation
    int Dim;
    int maxdim;
    T * mat;

void mcopy(const T* Bmat) // Bmat!=this->mat
{int i; T *ap; const T *bp;
  for(i=(Dim*(Dim+1))>>1,ap=this->mat,bp=Bmat ;i;
  i--,*(ap++)=*(bp++));}

public:
    // Konstruktoren
    sym_matrix(int vdim=0):Dim(vdim<0? 0:vdim),maxdim(Dim),
        mat((Dim>0)? new T[(maxdim*(maxdim+1))>>1]:(T*)NULL)
        {if(Dim>0 && mat== NULL)
            {cout <<"new failed in sym_matrix<T> constructor\n";exit(-1);}}

    sym_matrix(const sym_matrix<T>& B);/*:
        Dim(B.Dim),maxdim(B.Dim),mat((Dim>0)? new
T[(maxdim*(maxdim+1))>>1]:NULL)
        {if(&B==this){cout<<"initialisation from oneself not possible\n";
exit(-1);}
        mcopy(B.mat);}
    */
#ifdef math_types_h
    sym_matrix(const smatrix& B); // copy from smatrix
    sym_matrix<T>& operator=(const smatrix& B); // " "
#endif
    ~sym_matrix(){delete[]mat;Dim=maxdim=0;mat=NULL;}
    // destructor, leaves behind "empty" matrix

    sym_matrix<T>& operator=(const sym_matrix<T>& B);
    sym_matrix<T>& suck(sym_matrix<T>& B)
// flat copy, leaves f empty
    {if(&B!=this)
        {if(mat!=NULL) delete[]mat;
        mat=B.mat;Dim=B.Dim;maxdim=B.maxdim;B.mat=NULL;B.Dim=B.maxdim=0;}
        return *this;}

    T& coeff(int i,int j)
    {int h;if(i>j){h=i;i=j;j=h;}
    return this->mat[(j*(j+1)>>1)+i];}

    T coeff(int i,int j)const
    {int h;if(i>j){h=i;i=j;j=h;}
    return this->mat[(j*(j+1)>>1)+i];}

    int principal_minor(int i){if(i<=maxdim){Dim=i;return 1;}
    else return 0;}

    int dimension()const{return Dim;}
    int dim()const{return Dim;}

```

```

int max_dimension()const{return maxdim;}
int max_dim()const{return maxdim;}

// operators
sym_matrix<T>& operator*=(const T& n);

friend int compare<>(const sym_matrix<T>& A,const sym_matrix<T>& B);
friend void ijvertausch<>(sym_matrix<T>& A,int ii,int jj);
friend void ijdiag0add<>(sym_matrix<T>& A,register int ii,
register int jj);
friend int transform<>(sym_matrix<T>& A);
friend void make_primitive<>(sym_matrix<T>& A,T &factor);
friend int jacobistep<>(sym_matrix<T>& A,T& z,T& n,T& d);
friend void jacobidet<>(sym_matrix<T>& f,T& det,int* dimplus,
                        int* dimnull,int preserve_matrix);
/*inserted <> into jacobidet, RSP Jan 2009*/
};          // end sym_matrix<T>

```

I do not know whether there are more problems of the same type along the way.

Routines for big integers: This is presently a rather serious problem. Schiemann found it convenient to use some routines from LEDA for handling big integers, big floats, and for sorting. As an alternative he provided switches to use LiDIA instead. At the time Schiemann wrote his programs, LEDA was freely available in the scientific domain (though otherwise already a commercial product), meanwhile it is totally commercial and of no use for developing free software. LiDIA can still be downloaded but is no longer maintained. Consequently someone has to rearrange the routines that involve bigints and bigfloats; this goes definitely beyond my powers and time constraints. The problem is particularly bad for **hn** and for **isolist**. At present, both programs produce endless error messages and it is hard to distinguish which ones are caused by missing LEDA/LiDIA support and which ones are due to changed programming style alone.

Let us look at the LEDA/LiDIA problem more closely: In the Copyright file of the Hn-directory Schiemann writes:

```

Depending on the switches in the file 'include_param.h',
this program uses the integer interface to GNU's gmp
from LiDIA and the LiDIA type bigfloat.
The common setting is to prefer the data types
'integer' and 'bigfloat' from LEDA instead of
LiDIAs 'bigint','bigfloat'. (LiDIAs bigints need
a lot of time for copying, because they do not follow
a 'handle' concept as LEDAs integers.
With small numbers, LEDAs integers need about 1/10th
of the time compared with LiDIA. But that may depend
on the installation of LiDIA/LEDA on your system)

```

In addition, he uses the LEDA data type `sortseq` for sorting his lists. Information on these types can be obtained from the LEDA websites.

4. FURTHER GOALS

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