FINAL (Key)
JONES, Fall 1994

Solutions are only sketched for reasons of time (I did these in an hour plus!), apologies in advance for any typos and errors and such...

Problem 1. *Which of the following curves best represents the solution to the initial value problem* $dy/dx = x - y^3$*,* $y(0) = 1$*...*

SOLUTION. Since $y(0) = 1$, we have $(dy/dx)|_{x=0} = 0 - 1^3 = -1 < 0$, so the slope at the point $(0, 1)$ should be negative. That means that it can only be (a), (d), or (e). Notice also that when $x < 0$ and $y > 0$, we have $dy/dx < 0$, so for the points in the upper-left quadrant, the slope should be negative. This means that the solution is (d).

Problem 2. $\int e^{3x} \sin 2x \, dx$ *is...*

SOLUTION. The answer is (e). Integrate by parts twice:

$$\begin{aligned}\int e^{3x} \sin 2x \, dx &= \frac{1}{3} e^{3x} \sin 2x - \int \frac{1}{3} e^{3x} (2 \cos 2x) \, dx \\ &= \frac{1}{3} e^{3x} \sin 2x - \frac{2}{3} \left(\frac{1}{3} e^{3x} \cos 2x - \int \frac{1}{3} e^{3x} (-2 \sin 2x) \, dx \right).\end{aligned}$$

Now bring the integral to the other side.

Problem 3. *Suppose* a_n *is a sequence and* $\lim_{n\to\infty} a_n = \infty$*. Which of the following is true...*

SOLUTION. The answer is (a). By definition, $\lim_{n\to\infty} a_n = \infty$ if and only for every $M > 0$, there exists an N such that for $n \geq N$ we have $a_n \geq M$. This is exactly statement (a) with $M = 1/\epsilon$, if you do a bit of rearranging.

Problem 4. *Which of the following statements is false...*

SOLUTION. Note: the "O" in ODE stands for ordinary. It means nothing other than y is a simple function of x and no other variables.

Statement (c) is false: guessing $y(x) = e^{rx}$ to get the characteristic equation $ar^2+br+c = 0$ gives us the solutions to the *homogeneous* equation, and we have a nonhomogeneous equation (involving $G(x)$).

Problem 5. *In the method of variation of parameters one looks for a solution...*

SOLUTION. The answer is (d).

Problem 6. *Which of the following shaded regions...*

Solution. One way to solve this problem is to just check if certain values are in the set or not. We first test 0: since $|0 - i - 1| = |-i - 1| = \sqrt{2} \geq |0 + i| = 1$, zero is in the set, which means it must be (b), (d), or (e). We now test i: we get $|i - i - 1| = 1 \not\geq |i + i| = 2$, so i is not in the set, so it must be (b).

Problem 7. *The MacLaurin series for* $\ln(1 - x^3)$ *is...*

Solution. We have $\ln(1 - x) = -\sum_{n=1}^{\infty} x^n/n$, so

$$\ln(1 - x^3) = -\sum_{n=1}^{\infty} \frac{x^{3n}}{n}.$$

The answer is (b).

Problem 8. *Which of the following expressions gives the area...*

Solution. The only thing close is (d).

Problem 9. *Suppose the series* $\sum_{n=1}^{\infty}(-1)^n a_n$ *converges...*

Solution. Consider the power series $\sum_{n=1}^{\infty} a_n x^n$. This series has radius of convergence ≥ 1, since it converges for $x = -1$. Therefore it converges for $x = 1/3$, hence (a) is true.

Problem 10. *Which of the following cannot be integrated in terms of elementary functions...*

Solution. The answer is (b). Substitute $u = x + 1$, you end up with an integral $\int e^{u^2}\, du$ which fails.

Problem 11. *Let* f *be a function which is continuous...*

Solution. The answer is (c). Note we are not given that f is continuous at $x = 1$, so we have to take limits, it might also be improper there.

Problem 12. *Which of the following statements is correct concerning the complex number* $z = re^{i\theta}$*...*

Solution. The answer is (c), this is De Moivre's theorem: $z^5 = (re^{i\theta})^5 = r^5 e^{5i\theta} = r^5(\cos 5\theta + i \sin 5\theta)$.

Problem 13. *Find the general solution of* $dy/dx + e^{-y} + 1/x = 0$.

Solution. We substitute $y = \ln v$ and get $dy/dx = (1/v)(dv/dx)$, $e^{-y} = 1/v$, and

$$\frac{dv}{dx} + \frac{1}{x}v = -1.$$

The integrating factor is $I(x) = x$, so

$$d(xv) = -x\,dx$$

Integrating gives $v(x) = -x/2 + C/x$, so $y(x) = \ln(-x/2 + C/x)$.

Problem 14. *Find the solution of* $d^2y/dx^2 = xy$ *with* $y(0) = 1$, $dy/dx(0) = 0$.

Solution. This is a second-order equation with nonconstant coefficients, so we use the method of series. We get $y'' - xy = 0$, or

$$\begin{aligned} 0 &= \sum_{n=0}(n+1)(n+2)c_{n+2}x^n - \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= (2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 \dots) - (c_0x + c_1x^2 + c_2x^3 + \dots) \end{aligned}$$

We write out coefficients: we see that $c_2 = 0$, $c_3 = c_0/6$, $c_4 = c_1/12$, $c_5 = c_2/20 = 0$, and in general, $c_{n+2} = c_{n-1}/(n+1)(n+2)$. Note that $y(0) = c_0 = 1$ and $y'(0) = c_1 = 0$, so we need only consider coefficients that are multiples of 3. Therefore

$$c_n = \frac{c_{n-3}}{n(n-1)} = \frac{1}{n(n-1)(n-3)(n-4)\cdots 3\cdot 2}c_0$$

so since 3 divides n,

$$c_{3n} = \frac{1}{3n(3n-1)(3n-3)(3n-4)\cdots 3\cdot 2}$$

and

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{3n(3n-1)(3n-3)(3n-4)\cdots 3\cdot 2}x^{3n} = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \dots.$$

Problem 15. *Find the general solution to*

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \frac{1}{1+e^x}.$$

Solution. The complementary equation $y'' + 3y' + 2y = 0$ has characteristic equation $r^2 + 3r + 2 = (r+1)(r+2) = 0$, so $y_h(x) = c_1e^{-x} + c_2e^{-2x}$. We must now use the method of undetermined coefficients to solve the problem. We guess $y_p(x) = u_1(x)e^{-x} + u_2(x)e^{-2x}$, taking $y_1(x) = e^{-x}$, $y_2(x) = e^{-2x}$. Therefore we have the two equations

$$u_1'y_1 + u_2'y_2 = u_1'e^{-x} + u_2'e^{-2x} = 0$$

and

$$u_1'y_1' + u_2'y_2' = -u_1'e^{-x} - 2u_2'e^{-2x} = \frac{1}{1+e^x}.$$

Adding the two, we get

$$u_2'(x) = -\frac{e^{2x}}{1+e^x}.$$

Now integrate: substitute $v = e^x$ to get $dv = e^x\,dx = v\,dx$, and then you have

$$-\int \frac{v}{v+1}\,dv = -v + \ln|v+1| = -e^x + \ln|e^x+1|$$

by long division. By the first equation,

$$u_1'(x) = -e^{-x}u_2' = \frac{e^x}{1+e^x}$$

so by the same substitution,

$$u_1(x) = \int \frac{1}{1+v}\,dv = \ln|e^x+1|.$$

Therefore:

$$y(x) = y_h(x) + y_p(x) = c_1e^{-x} + c_2e^{-2x} + \ln|e^x+1|e^{-x} + (-e^x + \ln|e^x+1|)e^{-2x}.$$

Problem 16. *Sketch the field of tangents...*

SOLUTION. He promised no problems involving direction fields for the final!

Problem 17. *For what set of complex numbers...*

SOLUTION. We can treat convergence for complex numbers much like we can for real numbers. We can, in fact, use even the ratio test. For (i), we find $|z| < 1$, and for (ii) we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2+3}{n^2+3}\frac{n^n}{(n+1)^{n+1}}|z|.$$

Note that the first term tends to 1 as $n \to \infty$, and

$$\frac{n^n}{(n+1)^{n+1}} = \frac{1}{n+1}\frac{1}{(1+1/n)^n} = \frac{1}{n+1}\frac{1}{e} \to 0$$

as $n \to \infty$, so in fact the series converges for all complex numbers z.

Problem 18. *Find a family of orthogonal trajectories for the family of curves $y = c/x$.*

SOLUTION. The slope of the above curve is $-c/x^2$, so we want to solve $dy/dx = x^2/c$. Since $c = xy$, we want to solve $dy/dx = x/y$. This is separable: $y\,dy = x\,dx$ gives $y^2 = x^2 + C$.

Problem 19. *Give an example of each of the following...*

SOLUTION. For (1), we can take $\sum_{n=0}^{\infty} 1/n!$, which converges (to e, look at the power series). For (2), we can take $\sum_{n=0}^{\infty}(1/2)^n$, which converges (to 2, a geometric series). For (3), we can take $a_n = (-1)^n$, for example, or $a_n = -n$. For (4), take anything like $(x^2+1)/x$. For (5), we take something like $y'' + y = \sin x$.

Problem 20. *Evaluate each of the following integrals...*

SOLUTION. For (i), use integration by parts, taking $u = (\ln x)^2$, $dv = dx$, then repeat. You get:

$$\begin{aligned}\int (\ln x)^2\,dx &= x(\ln x)^2 - \int x(2\ln x)(1/x)\,dx = x(\ln x)^2 - 2(x\ln x - x) + C \\ &= x(\ln x)^2 - 2x\ln x + 2x + C.\end{aligned}$$

For (ii), substitute $\csc^2 x = 1 + \cot^2 x$ to get

$$\int \csc^3 x\,dx = \int \csc x\,dx + \int \cot^2 x \csc x\,dx.$$

The first integral is $\ln|\csc x - \cot x|$. The second can be done through integration by parts, taking $u = \cot x$, $dv = \csc x \cot x$, and

$$\int \cot^2 x \csc x\,dx = -\csc x \cot x - \int \csc^3 x\,dx.$$

Now add these back: you get

$$\int \csc^3 x\,dx = \frac{1}{2}\left(\ln|\csc x - \cot x| - \csc x \cot x\right) + C.$$

For (iii), make the trig substitution $x = 3\sin\theta$ to get

$$\int x^2\sqrt{9-x^2}\,dx = \int (3\sin\theta)^2(3\cos\theta)(3\cos\theta\,d\theta) = 81\int \cos^2\theta\sin^2\theta\,d\theta.$$

Now we note that $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$, so this becomes

$$81\int (1/2\sin(2\theta))^2\,d\theta = \frac{81}{4}\int \sin^2(2\theta)\,d\theta.$$

Now we substitute $\sin^2(2\theta) = (1-\sin(4\theta))/2$ (double the double-angle formula!) to get

$$\frac{81}{4}\int \frac{1-\sin(4\theta)}{2}\,d\theta = \frac{81}{8}\left(\theta + \cos(4\theta)\right) + C.$$

Now replace the limits of integration, to get

$$\int_0^3 x^2\sqrt{9-x^2}\,dx = \frac{81}{8}\left(\theta + \cos(4\theta)\right)_0^{\pi/2} = \frac{81}{8}(\pi/2 + 1 - 1) = \frac{81\pi}{16}.$$

For (iv) just use partial fractions:

$$\frac{x}{(x+3)(2x-7)} = \frac{3/13}{x+3} + \frac{7/13}{2x-7}$$

so

$$\int \frac{x}{(x+3)(2x+7)}\,dx = \frac{3}{13}\ln|x+3| + \frac{7}{26}\ln|2x+7| + C.$$