MIDTERM #2 (Key)
JONES, SPRING 1998

Problem 1. *Consider the series* $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n}$. *Which of the following is correct?*

(a) *The series converges absolutely.*
(b) *The series converges but not absolutely.*
(c) *The series does not converge because the nth term does not tend to zero.*
(d) *The series does not converge, by the ratio test.*
(e) *The series does not converge, by comparison with* $\sum_{n=1}^{\infty} 1/n$.

SOLUTION. Notice that $\cos(\pi/3) = 1/2$, $\cos(2\pi/3) = -1/2$, $\cos(3\pi/3) = -1$, $\cos(4\pi/3) = -1/2$, $\cos(5\pi/3) = 1/2$, $\cos(6\pi/3) = 1$, and then it repeats like this: therefore the series looks like

$$\frac{1}{2} - \frac{1}{2\cdot 2} - \frac{1}{3} - \frac{1}{2\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{6} + \frac{1}{2\cdot 7} - \frac{1}{2\cdot 8} - \frac{1}{9} - \frac{1}{2\cdot 10} + \frac{1}{2\cdot 11} + \frac{1}{12} + \dots.$$

Therefore statement (a) is false, because clearly

$$\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi/3)}{n}\right| \geq \sum_{n=1}^{\infty} \frac{1}{2n}$$

and the latter diverges as it is half of the harmonic series, which is divergent. Statement (c) is false, clearly the terms tend to zero. In (d), the ratio test does not apply, because

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\cos((n+1)\pi/3)}{\cos(n\pi/3)}\right| \frac{n}{n+1}$$

does not exist (notice the oscillation above). Statement (e) is false: we have that our series is smaller than $\sum_n 1/n$, which diverges, so we can conclude nothing.

Therefore the answer is (b). To see this, we should combine all of the negative terms together and all of the positive terms together to get an alternating series; without even computing the formulae, it is clear that resulting terms will be decreasing, alternatively positive and negative, and will tend to zero, so it is (conditionally) convergent by the alternating series test.

Problem 2. *Suppose numbers* a_n, *for* $n \geq 0$, *are given such that* $\sum_{n=0}^{\infty} 2^n a_n$ *converges. Which of the following is a consequence?*

(a) $\lim_{n\to\infty} a_n = 0$
(b) $\lim_{n\to\infty} a_n = 1$

(c) $\sum_{n=0}^{\infty} a_n/2^n$ *converges.*
(d) $\sum_{n=0}^{\infty} a_n$ *diverges.*
(e) $\lim_{k\to\infty} \left(\sum_{n=0}^{k} 2^n a_n\right) = 0.$

Solution. Statement (b) is false, since if this is true then

$$\lim_{n\to\infty} 2^n a_n = \lim_{n\to\infty 2^n} = \infty \neq 0$$

so the series could not converge. Statement (d) is false: a counterexample is $a_n = (1/4)^n$. Statement (e) is false: it reads that the limit of the partial sums tend to zero, so the sum would tend to zero, which we cannot necessarily conclude.

As far as I can tell, both (a) and (c) are true. Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

According to the big theorem about power series, either this converges only when $x = 0$, the series converges for all x, or there is a positive number R such that the series converges for $|x| < R$ and diverges for $|x| > R$. Since when $x = 2$ the series converges, we must have either the second or third case (with $R \geq 2$). This implies that the series converges for $x = 1$, i.e.

$$\sum_{n=0}^{\infty} a_n$$

converges. By the test for diverges, this implies that $\lim_{n\to\infty} a_n = 0$. Also, this implies that the series converges for $x = 1/2$, i.e.

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n}$$

converges.

I must be missing something here...

Problem 3. *The MacLaurin series for* $\sin^2 x$ *is*

(a) $x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^14}{7!} + \dots$
(b) $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$
(c) $x + \dfrac{x^5}{5!} + \dfrac{x^9}{9!} + \dfrac{x^{13}}{13!} + \dots$
(d) $1 - x + \dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \dots$
(e) $\dfrac{1}{2}\left(\dfrac{4x^2}{2!} - \dfrac{16x^4}{4!} + \dfrac{64x^6}{6!} - \dfrac{256x^8}{8!} + \dots\right)$

Solution. We use the identity

$$\sin^2 x = \frac{1-\cos(2x)}{2}.$$

We have

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

so

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \dots.$$

The answer is (e).

Problem 4. *Which of the following is an example of a sequence with* $\lim_n a_n = -\infty$*?*

(a) $a_n = (-1)^n n$.
(b) $a_n = (-1)^{n^2} n^2$.
(c) $a_n = \dfrac{e^n - e^{n^2}}{n}$.
(d) $a_n = \dfrac{n\ln(n+5)}{(n+1)(n+5)}$.
(e) $a_n = \cos n$.

Solution. To tend to $-\infty$ is to say that the terms grow more and more negative: more formally, for each negative number $-M$ ($M > 0$), we must have an N such that $a_n < -M$ for $N \geq n$. In words, this means that all of the terms after a certain point must be at least as negative as $-M$ for any negative number.

This is not true of (a), since half of the time the numbers are positive. The same is true for (b), as n^2 is alternatively even and odd. Series (d) fails because $\ln(n+5) > 0$ for sufficiently large n, so the sequence is actually positive eventually. And (e) fails because it oscillates between -1 and 1, so certainly cannot tend to $-\infty$. Therefore the answer is (c), and you can see this because $e^{n^2} > e^n$ for sufficiently large n, and this difference $e^{n^2} - e^n$ grows larger and larger as $n \to \infty$, even with the n in the denominator—to convince yourself of this, use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{e^x - e^{x^2}}{x} = \lim_{x\to\infty} \frac{e^x - 2xe^{x^2}}{1} = \lim_{x\to\infty} e^x(1 - 2xe^{x^2-x}) = -\infty.$$

So the terms a_n grow more and more negative, and indeed $\lim_{n\to\infty} a_n = -\infty$.

Problem 5. *Let* $J_0(x)$ *be the Bessel function*

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Which of the following is correct?

(a) $x^2\dfrac{d^2J_0}{dx^2}+x\dfrac{dJ_0}{dx}+x^2J_0=0.$
(b) $J_0(x)=e^x\cos(\ln x).$
(c) $\lim_{x\to 0} J_0(x)=2.$
(d) $\displaystyle\int_0^t J_0(x)\,dx=J_0(t).$
(e) $\dfrac{dJ_0}{dx}=J_0.$

SOLUTION. The answer is (a), this was homework problem #35, §11.9. You can verify it directly from the power series expansion.

To show that the others are false, you can see clearly by writing out the first few terms that (b) is false (otherwise, why would we bother defining a new function $J_0(x)$!?). Looking at the series we have $J_0(0)=1\neq 2$. Statement (d) would violate the fundamental theorem of calculus, and statement (e) is obviously false just by differentiating the first few terms.

Problem 6(a). *Find the MacLaurin series for*

$$\frac{x^2}{(1-x)^2}.$$

SOLUTION. We start with the series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n.$$

Differentiating, which we may do term-by-term within the radius of convergence $|x|<1$, we get

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}nx^{n-1}.$$

Therefore

$$\frac{x^2}{(1-x)^2}=\sum_{n=0}^{\infty}nx^{n+1}.$$

One can also do this by binomial series, requiring a lot more work.

Problem 6(b). *Find the sum of the series*

$$\sum_{n=2}^{\infty}n\left(\frac{1}{2}\right)^n.$$

SOLUTION. From the argument above, we notice that

$$\sum_{n=0}^{\infty}nx^n=\frac{x}{(1-x)^2}.$$

So

$$\sum_{n=0}^{\infty} n(1/2)^n = \frac{1/2}{1/4} = 2 = 0 + 1/2 + \sum_{n=2}^{\infty} n(1/2)^n.$$

So the sum is $3/2$.

Problem 7. *Consider the series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots.$$

What is the last term of the series that would have to be included to obtain $\pi/4$ *within* 0.01*? Justify your answer.*

SOLUTION. We use the alternating series remainder theorem, which states that for an alternating series

$$\sum_{n=0}^{\infty} (-1)^n b_n$$

with $0 \le b_{n+1} \le b_n$ and $b_n \to 0$ as $n \to \infty$, we have

$$\sum_{n=0}^{\infty} (-1)^n b_n \approx \sum_{n=0}^{N} (-1)^n b_n = s_N$$

with error $\le b_{N+1}$. That is to say,

$$|\pi/4 - s_N| \le b_{N+1}$$

which is

$$-b_{N+1} \le \pi/4 - s_N \le b_{N+1}$$

or

$$s_N - b_{N+1} \le \pi/4 \le s_N + b_{N+1}.$$

We need to find a b_{N+1} such that these two extremes agree to two decimal places (note it is not enough to insist $b_{N+1} \le 0.01$, since then this error will still alter the second decimal place!). It suffices to take $b_{N+1} \le 0.005$ since then this will no longer affect the second decimal place. Since $b_N = 1/(2N+1)$, $b_{N+1} = 1/(2(N+1)+3) = 1/(2N+3) \le 1/200$, we need to take $N \ge 99$.

Problem 8(i). *For the following series, say whether it converges absolutely, conditionally, or diverges. Give reasons:*

$$\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+50}.$$

SOLUTION. This series fails the test for divergence. Notice that

$$\lim_{n\to\infty} (-1)^n \frac{n}{n+50} \not\to 0$$

since, e.g. $n/(n+50) \to 1$. Therefore the series is divergent.

Problem 8(ii). *For the following series, say whether it converges absolutely, conditionally, or diverges. Give reasons:*

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}.$$

SOLUTION. The integral test applies: the function $f(x) = (\ln x)/x^2$ is continuous (for $x \geq 1$), it is positive, and decreasing, since

$$f'(x) = \frac{x^2(1/x) - 2x(\ln x)}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3},$$

and for $x \geq 1$, $x^3 \geq 0$ and $2\ln x < 1$.

Therefore the series converges or diverges as the integral

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx$$

converges or diverges.

We compute the indefinite integral

$$\int \frac{\ln x}{x^2}\, dx.$$

We use integration by parts. Let $u = \ln x$, so $du = 1/x\, dx$, and $dv = 1/x^2\, dx$, so $v = -1/x$, hence

$$\int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} - \int \frac{-1}{x^2}\, dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Hence

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} -\frac{\ln x}{x} - \frac{1}{x}\bigg|_1^t = \lim_{t\to\infty} -\frac{\ln t}{t} - \frac{1}{t} - (-1) = 0 + 1 = 1.$$

Therefore the series is convergent.