MATH 251: ABSTRACT ALGEBRA I
EXAM #2

Problem 1(a). Let G be a group and N a normal subgroup. What is the identity in the quotient group G/N?

Solution. We have $G/N = \{xN : x \in G\}$ with identity $1 \cdot N = N$.

Problem 1(b). Let $\sigma \in S_n$ be the product of m disjoint cycles of length k. Give necessary and sufficient conditions on m and k which determine when σ is an even permutation.

Solution. A permutation σ is even if and only if the number of even length cycles is even. Therefore σ is even if and only if k is odd or m is even.

Problem 1(c). True or false: Every subgroup of index 3 is normal. Give a proof or a counterexample.

Solution. False. The subgroup $\langle(1\ 2)\rangle \subset S_3$ is not normal, and $[S_3 : \langle(1\ 2)\rangle] = 6/2 = 3$.

Problem 2. Let $G = GL_2(\mathbb{Q})$ and let H be the subgroup

$$H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Q} \right\} \leq G.$$

Show that H is *not* a normal subgroup of G.

Solution. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $A \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. Thus $AHA^{-1} \not\subset H$, so H is not normal.

Problem 3. Let $\sigma = (1\ 2\ 4\ 5)(3\ 6)(8\ 9\ 10) \in S_{10}$, and let n be the order of σ. For which integers m with $0 \leq m \leq n$ is σ^m conjugate to σ? For each such m, exhibit a τ such that $\tau\sigma\tau^{-1} = \sigma^m$.

Solution. We have $n = \operatorname{lcm}(4, 2, 3) = 12$. The cycle type of σ is $(4, 2, 3)$, and we have that $\sigma^m = (1\ 2\ 4\ 5)^m(3\ 6)^m(8\ 9\ 10)^m$ has the same cycle type if and only if $(1\ 2\ 4\ 5)^m$ is a 4-cycle, and so on, if and only if $\gcd(m, 12) = 1$, which holds only for $m = 1, 5, 7, 11$. For $m = 1$ we have $\tau = ()$, and we have

$$\sigma^5 = (1\ 2\ 4\ 5)(3\ 6)(8\ 10\ 9), \quad \sigma^7 = (1\ 5\ 4\ 2)(3\ 6)(8\ 9\ 10), \quad \sigma^{11} = (1\ 5\ 4\ 2)(3\ 6)(8\ 10\ 9)$$

so we take $\tau = (9\ 10), (2\ 5), (2\ 5)(9\ 10)$, respectively.

Problem 4. Let $d, n \in \mathbb{Z}$ with $n > 1$, and suppose that $d \mid n$. Define

$$\begin{aligned} \phi : \mathbb{Z}/n\mathbb{Z} &\to \mathbb{Z}/n\mathbb{Z} \\ a &\mapsto \frac{n}{d} \cdot a \end{aligned}$$

(a) Show that the kernel of ϕ is $\ker(\phi) = \langle d \rangle$.
(b) Show that the image of ϕ is $\phi(\mathbb{Z}/n\mathbb{Z}) = \langle d \rangle \cong \mathbb{Z}/(n/d)\mathbb{Z}$.
(c) Deduce that $\mathbb{Z}/d\mathbb{Z} \cong \dfrac{\mathbb{Z}/n\mathbb{Z}}{\langle d \rangle}$.

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Solution. We have $a \in \ker(\phi)$ if and only if $(n/d)a \equiv 0 \pmod{n}$ if and only if $d \mid a$ if and only if $a \in \langle d \rangle$, which proves (a). For (b), we clearly have $\phi(\mathbb{Z}/n\mathbb{Z}) = \langle n/d \rangle$; now $\langle n/d \rangle$ is a cyclic group of order equal to the order of n/d, which is d, so $\langle n/d \rangle \cong \mathbb{Z}/d\mathbb{Z}$. Statement (c) follows directly from the First Isomorphism Theorem.

Problem 5. Let G be a group and let $N \trianglelefteq G$ a normal subgroup with $[G : N] = n$. Prove that for all $g \in G$, we have $g^n \in N$.

Solution. Consider the quotient G/N. We have $\#(G/N) = [G : N] = n$. By Lagrange's theorem, since $gN \in G/N$ we have $(gN)^n = g^n N = N$, which implies $g^n \in N$.