MATH 251: ABSTRACT ALGEBRA I
EXAM #2

Name ______________________________

Problem	Score
1	
2	
3	
4	
5	

Total __________

Date: 7 November 2007.

Problem 1.

(a) Let G be a group and N a normal subgroup. What is the identity in the quotient group G/N?

(b) Let $\sigma \in S_n$ be the product of m disjoint cycles of length k. Give necessary and sufficient conditions on m and k which determine when σ is an even permutation.

(c) True or false: Every subgroup of index 3 is normal. Give a proof or a counterexample.

Problem 2. Let $G = GL_2(\mathbb{Q})$ and let H be the subgroup

$$H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Q} \right\} \leq G.$$

Show that H is *not* a normal subgroup of G.

Problem 3. Let $\sigma = (1\ 2\ 4\ 5)(3\ 6)(8\ 9\ 10) \in S_{10}$, and let n be the order of σ. For which integers m with $0 \leq m \leq n$ is σ^m conjugate to σ? For each such m, exhibit a τ such that $\tau\sigma\tau^{-1} = \sigma^m$.

Problem 4. Let $d, n \in \mathbb{Z}$ with $n > 1$, and suppose that $d \mid n$. Define

$$\begin{aligned} \phi : \mathbb{Z}/n\mathbb{Z} &\to \mathbb{Z}/n\mathbb{Z} \\ a &\mapsto \frac{n}{d} \cdot a \end{aligned}$$

(a) Show that the kernel of ϕ is $\ker(\phi) = \langle d \rangle$.

(b) Show that the image of ϕ is $\phi(\mathbb{Z}/n\mathbb{Z}) = \left\langle \frac{n}{d} \right\rangle \cong \mathbb{Z}/d\mathbb{Z}$.

(c) Deduce that $\mathbb{Z}/d\mathbb{Z} \cong \dfrac{\mathbb{Z}/n\mathbb{Z}}{\langle d \rangle}$.

Problem 5. Let G be a group and let $N \trianglelefteq G$ a normal subgroup with $[G : N] = n$. Prove that for all $g \in G$, we have $g^n \in N$.