MATH 251: ABSTRACT ALGEBRA I
HOMEWORK #7

Problems (for all)

Problem 1 (sorta DF 3.2.7). Let $H \leq G$.

(a) Show that $aH = bH$ for $a, b \in G$ if and only if $b^{-1}a \in H$.
(b) Show that the relation $a \sim b \Leftrightarrow a^{-1}b \in H$ for $a, b \in G$ is an equivalence relation on G.

Problem 2 (DF 3.2.5). Let $H \leq G$.

(a) Let $g \in G$. Prove that $g^{-1}Hg$ is a subgroup of G having the same order as H.
(b) If $\#H = n$ and H is the unique subgroup of G of order n, deduce that $H \trianglelefteq G$.

Problem 3 (DF 3.2.6). Prove that if $H, K \leq G$ are finite subgroups of a group G whose orders are relatively prime then $H \cap K = \{1\}$.

Problem 4 (DF 3.2.12). Let $H \leq G$. Prove that the map $x \mapsto x^{-1}$ sends each left coset of H onto a right coset of H and gives a bijection between the set of left cosets and the set of right cosets of H.

Problem 5 (almost DF 3.2.16). Use Lagrange's theorem to prove *Fermat's little theorem*: If p is a prime, then $a^{p-1} \equiv 1 \pmod{p}$ for all $a \in \mathbb{Z}$ with $\gcd(a, p) = 1$.

Problem 6. Let G be a group.

(a) Show that the center $Z(G)$ is a normal subgroup of G.
(b) Show that if $G/Z(G)$ is cyclic, then G is abelian.

Problems (for grad students)

Problem 7. Let G be a group and $N \trianglelefteq G$ be a normal subgroup. Show that G/N is abelian if and only if $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

Problem 8 (DF 3.2.13–14).

(a) Fix any labelling of the vertices of a square. Use this to identify D_8 as a subgroup of S_4. Prove that the elements of D_8 and $\langle(1\ 2\ 3)\rangle$ do not commute in S_4.
(b) Prove that S_4 does not have a normal subgroup of order 8.

Date: 17 October 2007; due Wednesday, 24 October 2007.