MATH 251: ABSTRACT ALGEBRA I
REVIEW, EXAM #1

Problem 1. For each $a, b \in \mathbb{R}$ with $a \neq 0$, define the *linear map*

$$\begin{aligned} T_{a,b} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto T_{a,b}(x) = ax + b. \end{aligned}$$

Let G be the collection of all such linear maps, i.e. $G = \{T_{a,b} : a, b \in \mathbb{R}, a \neq 0\}$.

(a) Show that composition of linear maps defines a binary operation on G.
(b) Show that G is a group under composition. (You may assume that composition of maps is associative.)
(c) Prove that G is not abelian.

Problem 2. Let G be an abelian group.

(a) Let $a, b \in G$ have orders $2, 3$, respectively. What is the order of ab?
(b) Let $a, b \in G$ have orders $r, s \in \mathbb{Z}_{\geq 1}$ with $\gcd(r, s) = 1$. What is the order of ab?
(c) What can you say if G is *not* abelian?

Problem 3. Let G be an abelian group, and for $n \in \mathbb{Z}_{>0}$ let

$$G[n] = \{x \in G : x^n = 1\}.$$

Show that $G[n]$ is a subgroup of G.

Problem 4. Let G be a group and suppose that for all $x \in G$, we have $x^2 = 1$. Prove that G is abelian.

Problem 5. Let

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \subset M_3(\mathbb{R}).$$

(a) Show that H is a subgroup of $GL_3(\mathbb{R})$ (under matrix multiplication).
(b) Let $\mathbb{R}^2$ be a group under addition. Prove that the map

$$\begin{aligned} \phi : H &\to \mathbb{R}^2 \\ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} &\mapsto (a, c) \end{aligned}$$

is a homomorphism, but is not an isomorphism.

Problem 6. Let $\sigma = (1\ 4)(2\ 5\ 8\ 3\ 6\ 9) \in S_9$. Compute the order of σ^i for each integer $i \in \mathbb{Z}$.

Problem 7. Prove that the groups $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.

Problem 8. Let G be a finite group with $\#G = n > 2$. Show that there is no subgroup $H \leq G$ with $\#H = n - 1$.

Date: 5 October 2007; exam 8 October 2007.