MATH 20C: FUNDAMENTALS OF CALCULUS II
EXAM #3

Problem 1.

(a) Which one of the following functions is linear?

$$f(x,y,z) = \frac{2x+3y-5z}{7}$$
$$g(x,y,z,w) = x+y+z+w+xy+zw$$
$$h(x,y) = 5x+6y^2.$$

Solution. Only $f(x,y,z)$ is linear.

(b) Your weekly cost (in dollars) to manufacture x gallons of maple syrup and y pounds of maple butter is

$$C(x,y) = 1839 + 30x + 50y.$$

What is the marginal cost of a gallon of maple syrup? What does the slice $y =$ constant represent?

Solution. The marginal cost is $C_x = 30$ dollars per gallon (per week). The slice $y =$ constant represents the weekly cost in dollars to manufacture x gallons and a fixed amount of maple butter.

Problem 2. For the function

$$z = f(x,y) = 2\sqrt{x^2+y^2} - 9,$$

find the equation of the level curve where $z = -5$. Give a description of the graph of this curve.

Solution. We have $-5 = 2\sqrt{x^2+y^2} - 9$ or $4 = 2\sqrt{x^2+y^2}$ which after dividing by 2 and squaring yields $4 = x^2+y^2$: this is a circle centered at $(0,0)$ with radius 2.

Problem 3. Find the x-, y-, and z-intercepts of the function

$$z = f(x,y) = y^2 + 2xy + 4x^2 - 4.$$

Solution. To find the x-intercept, we set $y = z = 0$ to obtain $0 = 4x^2 - 4$ so $x = \pm 1$, so the x-intercepts are $(\pm 1, 0, 0)$. For the y-intercepts we set $x = z = 0$ to obtain $y^2 - 4 = 0$ so $y = \pm 2$ and hence they are $(0, \pm 2, 0)$. Similarly, we obtain the z-intercept as $(0,0,-4)$.

Problem 4. Label each graph below with the corresponding equation.

(a) $f(x,y) = e^{-(x^2+y^2)}$.
(b) $f(x,y) = x^2$.
(c) $f(x,y) = x+y+1$.
(d) $f(x,y) = x^2 - 2y^2$.

Solution. The answer is (b),(c),(d),(a).

Problem 5. Find the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ of the function

$$f(x,y) = \frac{1}{4x^2+3y-5xy}.$$

Solution. Writing $f(x,y) = (4x^2+3y-5xy)^{-1}$ we obtain from the chain rule:

$$f_x = -1(4x^2+3y-5xy)^{-2}(8x-5y) = \frac{5y-8x}{(4x^2+3y-5xy)^2}$$
$$f_y = -1(4x^2+3y-5xy)^{-2}(3-5x) = \frac{5x-3}{(4x^2+3y-5xy)^2}.$$

Problem 6. Find the partial derivatives $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x\,\partial y}, \dfrac{\partial^2 f}{\partial y^2}$ for
$$f(x,y) = e^{-2xy}.$$

Solution. We have $f_x = -2ye^{-2xy}$ and $f_y = -2xe^{-2xy}$. So $f_{xx} = 4y^2e^{-2xy}$ and $f_{yy} = 4x^2e^{-2xy}$, and by the product rule
$$f_{xy} = -2e^{-2xy} + (-2x)(-2y)e^{-2xy} = (4xy-2)e^{-2xy}.$$

Problem 7. Locate (but do not classify) all the critical points of the function
$$f(x,y) = xy + \frac{4}{x} + \frac{2}{y}.$$

Solution. Writing $f(x,y) = xy + 4x^{-1} + 2y^{-1}$ we obtain
$$f_x = y - 4x^{-2} = 0$$
$$f_y = x - 2y^{-2} = 0.$$

Solving for y in the first equation gives $y = 4x^{-2}$, and substituting this into the second equation yields $x - 2(4x^{-2})^{-2} = x - x^4/8 = 0$. Multiplying by -8 gives $x^4 - 8x = x(x^3-8) = 0$ so $x = 0$ or $x = 2$. Substituting back into $y = 4x^{-2}$ gives respectively y is undefined and $y = 1$, so the only critical point is $(2,1)$.

Problem 8. The function
$$f(x,y) = 2x^2 + y^2 - x^2y^2$$
has a critical point at $(0,0)$. Determine if this point is a relative maximum, relative minimum, or saddle point.

Solution. We have $f_x = 4x - 2xy^2$ and $f_y = 2y - 2x^2y$. So $f_{xx} = 4 - 2y^2$, $f_{xy} = -4xy$ and $f_{yy} = 2 - 2x^2$. Thus the Hessian is
$$H = (4-2y^2)(2-2x^2) - (-4xy)^2$$
so $H(0,0) = 8 > 0$ and $f_{xx}(0,0) = 4 > 0$ so the point is a relative minimum.