Problem 1.
(a) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for

$$f(x, y) = \frac{1}{4xy^2 + 3x + 1}.$$ 

Solution. We write $f(x, y) = (4xy^2 + 3x + 1)^{-1}$. Then

$$\frac{\partial f}{\partial x} = -(4y^2 + 3)(4xy^2 + 3x + 1)^{-2}$$

and

$$\frac{\partial f}{\partial y} = -(8xy)(4xy^2 + 3x + 1)^{-2}.$$

(b) Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$ for

$$f(x, y) = e^{xy}.$$ 

Solution. We have $\frac{\partial f}{\partial x} = ye^{xy}$ and $\frac{\partial f}{\partial y} = xe^{xy}$. So

$$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy} \quad \frac{\partial^2 f}{\partial x \partial y} = 2xe^{xy} \quad \frac{\partial^2 f}{\partial y^2} = x^2 e^{xy}$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} (xe^{xy}) = e^{xy} + x(ye^{xy}) = (1 + xy)e^{xy}.$$

Problem 2.
(a) Find all critical points of the function

$$f(x, y) = x^2 - xy^2 + \frac{1}{5}y^5.$$ 

Solution. We set

$$\frac{\partial f}{\partial x} = 2x - y^2 = 0$$

and

$$\frac{\partial f}{\partial y} = -2xy + y^4 = 0.$$ 

In the first equation, we can solve for $x$ to obtain $2x = y^2$ so $x = y^2/2$; substituting this into the second equation, we get

$$-2(y^2/2)y + y^4 = -y^3 + y^4 = 0$$

or equivalently

$$y^4 - y^3 = y^3(y - 1) = 0$$

so $y = 0, 1$. Substituting these into $x = y^2/2$ gives the critical points $(0, 0), (1/2, 1)$.

(b) Compute the Hessian

$$H = f_{xx}f_{yy} - f_{xy}^2.$$ 

Which of the two critical points is a local minimum?

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Solution. We compute that

\[ f_{xx} = 2 \]
\[ f_{yy} = -2x + 4y^3 \]
\[ f_{xy} = -2y \]

so

\[ H = 2(-2x + 4y^3) - (-2y)^2 = -4x + 8y^3 - 4y^2 \]

So \( H(0, 0) = 0 \) and so we cannot determine from the Hessian what kind of critical point it is; however, \( H(1/2, 1) = -2 + 8 - 4 = 2 > 0 \) and \( f_{xx} = 2 > 0 \) so the point \((1/2, 1)\) is a local minimum.