Problem 1. Decide if the integral converges, and if so compute its value.

(a) \( \int_0^{\infty} \frac{2x}{x^2 + 1} \, dx \).

Solution. By definition, \( \int_0^{\infty} \frac{2x}{x^2 + 1} \, dx = \lim_{M \to \infty} \int_0^M \frac{2x}{x^2 + 1} \, dx \).

We substitute \( u = x^2 + 1 \) so \( du = 2x \, dx \). When \( x = 0 \) we have \( u = 1 \) and when \( x = M \) we have \( u = M^2 + 1 \). Hence
\[
\lim_{M \to \infty} \int_0^M \frac{2x}{x^2 + 1} \, dx = \lim_{M \to \infty} \int_1^{M^2 + 1} \frac{du}{u} = \lim_{M \to \infty} \ln u \bigg|_1^{M^2 + 1}
\]
so the integral diverges.

(b) \( \int_{-\infty}^{-1} \frac{1}{x^{4/3}} \, dx \).

Solution. We have
\[
\int_{-\infty}^{-1} \frac{1}{x^{4/3}} \, dx = \lim_{M \to -\infty} \int_{M}^{-1} x^{-4/3} \, dx = \lim_{M \to -\infty} \frac{x^{-1/3}}{-1/3} \bigg|_{M}^{-1} = \lim_{M \to -\infty} \left( -\frac{3}{(-1)^{1/3}} + \frac{3}{M^{1/3}} \right)
\]
and the integral converges.

(c) \( \int_0^{2} \frac{1}{x^2} \, dx \).

Solution. Here, there is a discontinuity at \( x = 0 \), since \( 1/x^2 \to +\infty \) as \( x \to 0 \). So
\[
\int_0^{2} \frac{1}{x^2} \, dx = \lim_{a \to 0^+} \int_a^{2} x^{-2} \, dx = \lim_{a \to 0^+} -x^{-1} \bigg|_a^{2} = \lim_{a \to 0^+} \left( -\frac{1}{2} + \frac{1}{a} \right) = -\frac{1}{2} + \infty = \infty.
\]

Problem 2. Sales of the text Calculus and You have been declining continuously at a rate of 5% per year. Assuming that Calculus and You currently sells 5000 copies per year and that sales will continue this pattern of decline, calculate the total future sales of the text.

Solution. Let \( S(t) \) denote the rate of sales of the text in \( t \) years. Then \( S(t) = 5000e^{-0.05t} \) since sales are declining at an exponential rate, and therefore the total sales is
\[
\int_0^{\infty} S(t) \, dt = \lim_{M \to \infty} \int_0^M 5000e^{-0.05t} \, dt = \lim_{M \to \infty} 5000 \left( e^{-0.05M} - e^{-0.05} \right)
\]
\[
= \lim_{M \to \infty} -100000 \left( e^{-0.05M} - 1 \right) = -100000(0 - 1) = 100000.
\]

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