MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
IN CLASS FINAL REVIEW

Problem 1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f([a, b]) \subset [a, b]$. Show that f has a fixed point, i.e., there exists $x \in [a, b]$ such that $f(x) = x$.

Problem 2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions and define

$$h(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{Q}; \\ g(x), & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that h is continuous at c if and only if $f(c) = g(c)$. Assuming that f, g are differentiable, show that h is differentiable at c if and only if $f'(c) = g'(c)$.

Problem 3. If a_n is bounded and $b_n \to 0$ show that $\lim a_n/b_n$ does not exist.

Problem 4. Let $A, B \subset \mathbb{R}$ be bounded above, and suppose that $\sup A < \sup B$. Show that there exists $b \in B$ such that b is an upper bound for A.

Problem 5. Let $f(x)$ be a polynomial of odd degree. Show that for every $a \in \mathbb{R}$, there exists $c \in \mathbb{R}$ such that $f(c) = a$.

Problem 6. Prove that every sequence in $\mathbb{R}$ has a monotone subsequence.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $A \subset \mathbb{R}$ is compact. Is $f^{-1}(A)$ necessarily compact?

Problem 8. Use the mean value theorem to show that if f, f' are both strictly increasing functions then f is unbounded.

Problem 9. Prove using the definition of compact (that every sequence in K has a subsequence which converges in K) to show that every compact set has a maximum.

Problem 10. Show that the function $f(x) = \sqrt{|x|}$ is uniformly continuous on $\mathbb{R}$.

Problem 11. Consider the sequence of functions $f_n(x) = 1 + \frac{\sin(nx)}{n}$. Show that f_n converge uniformly to a function f and state f.

Problem 12. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions which converge uniformly to $f : [0, 1] \to \mathbb{R}$. Prove that f is uniformly continuous.

Problem 13. Is the set of all finite subsets of $\mathbb{N}$ countable or uncountable?

Problem 14. True or false: Let $A_1, A_2, \dots$ be compact subset of $\mathbb{R}$ such that $A = \bigcup_{n=1}^{\infty} A_n$ is compact. Then $A = \bigcup_{n=1}^{N} A_n$ for some $N \in \mathbb{N}$.

Problem 15. Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Problem 16. Let $f, g : A \to \mathbb{R}$ be uniformly continuous functions. Prove that $f - g$ is uniformly continuous on A, where $f - g : A \to \mathbb{R}$ is defined by $(f - g)(x) = f(x) - g(x)$.

Date: 9 December 2009; exam Tuesday, 15 December 2009, 3:30–6:30 p.m.

Problem 17. Is $f(x) = x|x|$ differentiable at $x = 0$?

Problem 18. Let F_n be the nth Fibonnaci number, where $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Show that the limit

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

exists and compute the value of this limit.

Problem 19. Let $a_n \geq 0$ and suppose that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n/n$ converges.

Problem 20. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

Show that h is continuous on $\mathbb{R}$. Is h differentiable? If so, is the derivative function h' continuous?

Problem 21. Let $f : A \to \mathbb{R}$ be a function. Suppose that there exists λ with $0 < \lambda < 1$ such that

$$|f(x) - f(y)| \leq \lambda |x - y|$$

for all $x, y \in A$. Show that f is uniformly continuous on A.

Problem 22. Prove that there exists $c \in (0, \pi)$ such that the line tangent to the graph of $f(x) = \sin x + x^3 - \pi^2 x$ at $(c, f(c))$ has slope zero.