# MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
# EXAM #2

**Problem 1**. Statement (a) is true: since $A$ is infinite, there is a sequence $(a_n)$ with $a_n \in A$ and all $a_n$ distinct; this sequence is bounded, so by Bolzano-Weierstrass, it has a convergent subsequence, and its limit is then a limit point of $A$. Statement (b) is false: what is true is $f^{-1}(U)$ is open whenever $U$ is open. As shown in class, a counterexample is $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x/(1 + x^2)$ which has $f(\mathbb{R}) = [0, 1/2]$. Statement (c) is true: every point of $S$ is an isolated point so $S$ has no limit points so $S$ is closed. Statement (d) is false: we need a strict inequality $f(a) < L < f(b)$ in order to ensure that $c \in (a, b)$; a counterexample is $f(x) = x^2$ which has $f(-1) = f(1) = 1$ but $f(c) \neq 1$ for all $c \in (-1, 1)$. Statement (e) is false: for example, $A = [0, 1) \cup (1, 2]$ is not compact since it is not closed, but $\max A = 2$ and $\min A = 0$.

**Problem 2**. For part (a), we treat two cases. First suppose $c = 0$. Let $\epsilon > 0$. Let $\delta = \epsilon^2$. Then, if $|x| < \delta$ then $|\sqrt{x}| < \epsilon$. Now assume $c > 0$. Let $\epsilon > 0$. Let $\delta = \epsilon\sqrt{c} > 0$. Then, if $|x - c| < \delta$ we have

$$|\sqrt{x} - \sqrt{c}| = \frac{|x - c|}{|\sqrt{x} + \sqrt{c}|} = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}} < \frac{\sqrt{c}\epsilon}{\sqrt{c}} = \epsilon.$$

For part (b), since $f$ is continuous on $[0, 1]$ which is compact, $f$ is uniformly continuous by our theorem. For part (c), we have $f'(x) = 1/(2\sqrt{x})$ defined on $\mathbb{R}_{>0}$. Consider the sequences $(x_n), (y_n)$ where $x_n = 1/n^2$ and $y_n = 1/(n + 1)^2$. Then $x_n, y_n \to 0$ so $|x_n - y_n| \to 0$ but $|f'(x_n) - f'(y_n)| = |n/2 - (n + 1)/2| = 1/2$. So $f'$ is *not* uniformly continuous on $\mathbb{R}_{>0}$ by the sequential criterion.

**Problem 3**. Let $\epsilon = L > 0$. Since $\lim_{x\to c} f(x) = L$ exists, there exists $\delta > 0$ such that $|f(x) - L| < L$ whenever $0 < |x - c| < \delta$ and $x \in A$. Let $U = V_\delta(c)$. Then, if $x \in U \cap A$ then $|f(x) - L| < L$ so $-L < f(x) - L < L$ and thus $0 < f(x)$ as desired.

**Problem 4**. For (a), we compute for $x > 0$ that

$$f'(x) = ax^{a-1} \sin(\log(x)) + x^{a-1} \cos(\log(x)) = x^{a-1}(a \sin(\log(x)) + \cos(\log(x)))$$

and $f'(x) = 0$ if $x < 0$.

For (b), we claim that $f$ is differentiable if (and only if) $a > 1$, in which case $f'(0) = 0$. We prove this for $a = 2$. We compute

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}.$$

Let $\epsilon > 0$. Let $\delta = \epsilon$. Then if $x \neq 0$ and $|x| < \delta$ then

$$\left|\frac{f(x)}{x}\right| = |x \sin(\log(x))| \leq |x| < \epsilon.$$

In fact, for $a = 2$ the derivative $f'$ is also continuous. We prove that $\lim_{x\to 0} f'(x) = 0$. Let $\epsilon > 0$, and let $\delta = \epsilon/3$. We claim if $|x| < \delta$ then $|f'(x) - f'(0)| = |f'(x)| < \epsilon$. If $x < 0$ then $f'(x) = 0$ already; if $x > 0$ then

$$|f'(x)| = |x||2 \sin(\log(x)) + \cos(\log(x))| \leq 3|x| < 3(\epsilon/3) = \epsilon.$$

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