MATH 251: ABSTRACT ALGEBRA I
EXAM #1

Name ________________________________

Problem	**Score**
1	
2	
3	
4	
5	

Total ____________

Date: 30 September 2011.

Problem 1.

(a) Show that $a = 52 \in (\mathbb{Z}/109\mathbb{Z})^\times$ by exhibiting $c \in \mathbb{Z}/109\mathbb{Z}$ with $ac \equiv 1 \pmod{109}$.

(b) Characterize the set of integers $n \geq 2$ such that the following statement is true:

$$\text{For all } a, b \in \mathbb{Z}/n\mathbb{Z}, \text{ if } ab = 0 \text{ then } a = 0 \text{ or } b = 0.$$

Give a proof that your answer is correct.

Problem 2. In each problem, justify your answer.

(a) True or false: The operation $\star$ on $\mathbb{Q}$ defined by $a \star b = \frac{a+b}{5}$ is associative and commutative.

(b) True or false: The set $\{x \in \mathbb{Q} : |x| < 1\}$ is a group under addition.

(c) In the group $D_{10} = \langle r, s \mid r^5 = s^2 = 1,\ rs = sr^{-1} \rangle$, find an element
$$x \in \{1, r, \ldots, r^4, s, sr^2, \ldots, sr^4\}$$
such that $x = r^2 s r^{-1} s^2 r^6$.

(d) True or false: The element $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ belongs to $\mathrm{GL}_2(\mathbb{F}_3)$.

(e) True or false: If G is a group under $\star$, then for each $a \in G$, the inverse a^{-1} is unique. If true, give a short proof; if false, give a counterexample.

Problem 3. Let G be a group, and let $g \in G$. Consider the map

$$\begin{aligned} \phi : G &\to G \\ x &\mapsto g^{-1}xg. \end{aligned}$$

(a) Show that ϕ is a homomorphism. What is the inverse of ϕ?

(b) Show that ϕ is the identity map if and only if g commutes with all elements of G. If ϕ is the identity map, does this mean that G is abelian?

Problem 4. Let σ be the permutation

$$\begin{array}{ccccc} 1 \mapsto 6 & 2 \mapsto 9 & 3 \mapsto 10 & 4 \mapsto 1 & 5 \mapsto 5 \\ 6 \mapsto 4 & 7 \mapsto 2 & 8 \mapsto 8 & 9 \mapsto 7 & 10 \mapsto 3 \end{array}$$

and let τ be the permutation

$$\begin{array}{ccccc} 1 \mapsto 1 & 2 \mapsto 5 & 3 \mapsto 9 & 4 \mapsto 4 & 5 \mapsto 3 \\ 6 \mapsto 6 & 7 \mapsto 8 & 8 \mapsto 2 & 9 \mapsto 7 & 10 \mapsto 10 \end{array}.$$

(a) Find the cycle decompositions of the permutations $\sigma, \tau, \sigma^2, \sigma\tau$.

(b) Compute the order of the permutations σ, τ.

(c) Does S_7 contain an element of order 8? If so, give such an element explicitly; if not, give a proof.

Problem 5. Show that the groups

$$\mathbb{Z}/8\mathbb{Z},\ D_8,\ Q_8, \text{ and } S_8$$

are mutually nonisomorphic groups.