MATH 251: ABSTRACT ALGEBRA I
EXAM #2

Problem 1(a). The order of $H_n = \langle x^n \rangle$ is $132/\gcd(n, 132)$. Since $\gcd(38, 132) = 2$ we have $H_{38} = H_2$ and since $\gcd(110, 132) = 22$ we have $H_{110} = H_{22}$. Since $2 \mid 22$, we have $H_{38} = H_2 = \langle x^2 \rangle \supset \langle x^{22} \rangle = H_{22} = H_{110}$.

Problem 1(b). We have $\#(G/H) = \#G/\#H = 117/9 = 13$. Thus G/H is a group of prime order, so it is cyclic, so it is abelian.

Problem 2(a). We have $\phi(1) \in \phi(N)$ so $\phi(N) \neq \emptyset$.

Let $x, u \in \phi(N)$. Then by definition, there exists $n, m \in N$ such that $x = \phi(n)$ and $u = \phi(m)$. Thus $xu = \phi(n)\phi(m) = \phi(nm)$ since ϕ is a homomorphism, and $nm \in N$ since $N \leq G$. Therefore $xu \in \phi(N)$.

Finally, let $x \in \phi(N)$. Then $x = \phi(n)$ for some $n \in N$. So $x^{-1} = \phi(n)^{-1} = \phi(n^{-1})$ and $n^{-1} \in N$, so $x^{-1} \in \phi(N)$.

Problem 2(b). Let $x = \phi(n) \in \phi(N)$ and let $h \in H$. Then, since ϕ is surjective, there exists $g \in G$ such that $\phi(g) = h$. Thus

$$hxh^{-1} = \phi(g)\phi(n)\phi(g)^{-1} = \phi(gng^{-1}).$$

Since $N \trianglelefteq G$, we have $gng^{-1} \in N$. Thus $hxh^{-1} \in \phi(N)$, so $\phi(N) \trianglelefteq H$.

Problem 3. If $a = 1$, then $1 \in Z(G)$, so we may assume $a \neq 1$.

Let $x \in G$. Since $\langle a \rangle \trianglelefteq G$, we have $xax^{-1} \in \langle a \rangle = \{1, a\}$. Therefore $xax^{-1} = 1$ or $xax^{-1} = a$. But in the former case, by cancellation we have $a = x^{-1}x = 1$, which we just excluded. So we have the second case, and $xa = ax$. Therefore $a \in Z(G)$.

Problem 4(a). The cycle lengths are $5, 3, 2$ so it is even + even + odd, so σ is odd.

Problem 4(b). Given as Example (3) in Section 2.5, page 69.

Problem 4(c). Since every element in A_n is a product of an even number of transpositions, it suffices to show that every product σ of two transpositions is the product of 3-cycles. If the two transpositions have an element in common, then without loss of generality it is of the form $\sigma = (a\ b)(b\ c) = (a\ b\ c)$ for $a, b, c \in \{1, \dots, n\}$ and so σ is already a 3-cycle. If the two transpositions are disjoint, then we have $(a\ b)(c\ d) = (a\ b)(b\ c)(b\ c)(c\ d) = (a\ b\ c)(b\ c\ d)$ is also the product of 3-cycles.

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