MATH 251: ABSTRACT ALGEBRA I
FINAL EXAM

Name ____________________________________

Problem	**Score**	**Problem**	**Score**
1		6	
2		7	
3		8	
4		9	
5		10	

Total ____________

Date: 16 December 2011.

Problem 1.

(a) List all solutions to $x^2 = 1$ in Q_8.

(b) Let $G = \langle x \rangle$ be a cyclic group of order 105. List the elements of order 7 in G.

(c) Are the groups $\mathbb{Z}/8\mathbb{Z}$ and $(\mathbb{Z}/16\mathbb{Z})^\times$ isomorphic? Explain.

(d) Let σ be the permutation

$$1 \mapsto 2 \quad 2 \mapsto 1 \quad 3 \mapsto 3 \quad 4 \mapsto 5 \quad 5 \mapsto 4 \quad 6 \mapsto 6$$

and let τ be the permutation

$$1 \mapsto 6 \quad 2 \mapsto 1 \quad 3 \mapsto 2 \quad 4 \mapsto 4 \quad 5 \mapsto 3 \quad 6 \mapsto 5.$$

Compute the cycle decomposition of $\rho = \sigma\tau^2$. Is ρ even or odd?

Problem 2.

(a) What is the largest order of an element in S_{10}?

(b) Let G be a group, and let $H = \{a \in G : a^2 = 1\}$. If G is abelian, show that $H \leq G$. Give an example of a nonabelian group G for which H is not a subgroup.

Problem 3.

(a) In a finite group G, show that the order of ab is equal to the order of ba for all $a, b \in G$.

(b) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G = \mathrm{GL}_2(\mathbb{R})$. Compute $C_G(A)$.

(c) List all ideals of $\mathbb{C}$.

Problem 4.

(a) Show that $\mathrm{Inn}(D_8)$ is abelian. What is its isomorphism type?

(b) Let R be a commutative ring. Show that

$$I = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in R \right\} \subseteq \mathrm{M}_2(R)$$

is a left ideal of $\mathrm{M}_2(R)$. Is it a two-sided ideal?

Problem 5.

(a) Let $\phi : R \to S$ be an isomorphism of rings. Carefully show that ϕ restricted to $R^\times$ yields an isomorphism $\phi : R^\times \to S^\times$ of groups.

(b) List all abelian groups of order 450 up to isomorphism, giving their invariant factors and elementary divisors.

Problem 6.

(a) Let R be an integral domain and $I \subsetneq R$ a proper ideal. Must R/I be an integral domain? If so, give a proof; if not, give a counterexample.

(b) Let $R = (\mathbb{Z}/13\mathbb{Z})[i] = \{x + yi : x, y \in \mathbb{Z}/13\mathbb{Z}\}$, with $i^2 = -1$. Find a zerodivisor in R.

Problem 7. Let G be a group and let $N \trianglelefteq G$. Show that G/N is abelian if and only if $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

Problem 8.

(a) Let G be a group of order 42. Show that G has a normal subgroup H of order 7. If $H \subseteq Z(G)$ and G/H is abelian, show that G is abelian.

(b) Let G be a group of order 105. Show that G has both a normal subgroup of order 5 and a normal subgroup of order 7.

Problem 9. Exhibit rigorously an isomorphism

$$G = \langle x, y, z : x^2 = y^3 = z^3 = xyz = 1 \rangle \xrightarrow{\sim} A_4.$$

You may assume that $\#G = 12$.

Problem 10. Let G be a group with $\#G = p^2q$ where p, q are primes and $p > q$. Let $H, K \leq G$ be distinct subgroups of order p^2.

(a) Show that $H \cap K$ is a subgroup of order p.

(b) Show that $N = H \cap K$ is normal in G.