MATH 052: INTRODUCTION TO PROOFS
HOMEWORK #11

Problem 2.3.1.

(a) $(\exists n \in \mathbb{N})(n + 15 = 22)$.
(b) $(\forall n \in \mathbb{N})(n^3 + 15 = 22)$.
(c) $\sim(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y^2 = x)$.
(d) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x \neq y^2)$.
(e) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y^3 = x)$.

Problem 2.3.2. Part (a) is the statement "If there exists x such that $x \in A$ and $x \in \mathbb{Z}$, then for all x we have $x \in A$ and $x \in \mathbb{Z}$". This implication is false, because the hypothesis is true ($x = 1 \in A$) but the conclusion is false ($3 \notin A$).

Part (b) is the statement "If for all x we have $x \in A$ and $x \in \mathbb{Z}$, then there exists x such that $x \in A$ and $x \in \mathbb{Z}$". This implication is true, because the hypothesis is false ($3 \notin A$).

Problem 2.3.3. For part (a), the first implication is now true, because the hypothesis is false (there does not exist an x such that $x \in \emptyset$), and the second implication is still true for the same reason. For part (b), no, there is no such set, because the statement "for all $x \in \mathbb{R}$ we have $x \in A$ and $x \in \mathbb{Z}$ is always false for any set A, so the implication is always true.

Problem 2.3.5. For part (a), we have:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x')(|x' - x| < \delta \Rightarrow |f(x') - f(x)| < \delta).$$

For part (b), we have

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x')(|x' - x| < \delta \wedge |f(x') - f(x)| \geq \delta).$$

In negating a nested statement like this, swap $\exists$ and $\forall$; and the negation of an implication $P \Rightarrow Q$ is the conjunction $P \wedge \sim Q$. For part (c), "A function f is not continuous at $x \in \mathbb{R}$ if there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x' \in \mathbb{R}$ such that $|x' - x| < \delta$ and $|f(x') - f(x)| \geq \delta$."

Problem 2.3.6. p is a prime number if and only if

$$(p \geq 2) \wedge (p = ab \Rightarrow (a = p \vee b = p)).$$

The best answer, which is not obvious at this point, is that p is prime if and only if $p \geq 2$ and

$$(p \mid ab) \implies (p \mid a \wedge p \mid b).$$

Problem 2.3.9.

(a) True: Every nonnegative real number has a square root.
(b) False: Every positive real number has two square roots.
(c) True: Every real number has a unique cube root.
(d) True: The unique real number x such that $xy = y$ for all y is $x = 1$.
(e) False: The element $x = 0$ has the property that all $y \in \mathbb{R}$ satisfy $xy = 0$.
(f) True: Take $x = 1$, then the only $y \in \mathbb{R}$ which satisfies $xy = 0$ is $y = 0$.
(g) True: We must have $y = -x$.
(h) True: $(\forall x)(\exists y)(x > y)$ is true, we can always find a real number larger than any given real number.