ORAL QUALIFYING EXAM QUESTIONS

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Below are some questions that I have asked on oral qualifying exams (starting in fall 2015).

1. Algebra

1.1. Core questions.

(1) Let R be a noetherian (commutative) domain, let M be a finitely generated R-module, and suppose $f : M \to M$ is a surjective R-module homomorphism. Show that f is injective—and then show that this conclusion fails if M is not finitely generated or if R is not noetherian.

(2) Consider the category of groups (under homomorphisms) and the category of sets (under maps). Define a map which associates to a group G the set $\{g \in G : g^2 = 1\}$. Show that this map is a functor. On the other hand, if we associate to G the set of elements of order 2, show that it is *not* a functor.

(3) Explain the proof that every PID is a UFD (in particular, why prime and irreducible are the same) and exhibit a UFD that is not a PID.

(4) Let S be the p-Sylow subgroup of $G = \mathrm{GL}_n(\mathbb{F}_q)$, where $q = p^a$. Compute $\#S$, decide when S is normal in G. Then show that there is a nonzero fixed point for the natural action of S on $\mathbb{F}_q^n$.

(5) Let ℓ be an odd prime. Describe the Sylow ℓ-subgroups of $G = \mathrm{SL}_2(\mathbb{F}_q)$ with $q = p^a$.

(6) Let R be a commutative ring and let S be a multiplicatively closed subset containing 1. Define the localization $R[S^{-1}]$, define a natural map $i_R : R \to R[S^{-1}]$, and explain the relationship between the prime ideals of R and the prime ideals of $R[S^{-1}]$. Give an example where i_R is not injective, and show that the association from R to $R[S^{-1}]$ is functorial in R (suitably interpreted).

1.2. Representation theory of finite groups.

(1) Construct explicitly the irreducible 2-dimensional representation of S_3 by considering the rigid motions of an equilateral triangle in $\mathbb{R}^2$.

(2) Construct the character table of the quaternion group Q_8 of order 8.

1.3. Integral extensions.

(1) Let A be a (commutative) domain. Define what it means for a ring B to be integral over A and give several equivalent formulations.

(2) Give a definition of Dedekind domain and several equivalent formulations. Is there an uncountable Dedekind domain?

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(3) Let L/K be a finite extension of fields and consider the trace pairing

$$\begin{aligned} L \times L &\to K \\ x, y &\mapsto \mathrm{Tr}_{L/K}(xy). \end{aligned}$$

Under what hypotheses is this pairing nondegenerate? What does the pairing look like when L is a purely inseparable extension of K?

(4) Let A be noetherian and integrally closed in its field of fractions K, let L be a finite separable extension of K, and let B be the integral closure of A in L. Show that B is finitely generated as an A-module.

2. Number theory

2.1. **Algebraic number theory: global theory.**

(1) Let $K = \mathbb{Q}(\sqrt{3})$. Describe its ring of integers, discriminant, and ramification and splitting of primes (in terms of congruences). Exhibit the smallest m such that K is a subfield of the cyclotomic field $L = \mathbb{Q}(\zeta_m)$, and then show that L/K is unramified at all (finite) primes of K—but that L is ramified over K at each of the infinite places. Does K have any everywhere unramified abelian extension?

(2) Give several equivalent definitions of a Dedekind domain. State which of the following are Dedekind domains:

$$k[[t]] \ (k \text{ a field}), \quad \{a/b \in \mathbb{Q} : 3 \nmid b\}, \quad \mathbb{C}[x,y]/(y^2 - x^3).$$

(3) A Dedekind domain is noetherian, integrally closed, and Krull dimension 1. Give examples of domains that satisfy exactly two of the three (but not the third) in each of the three cases.

(4) Compute as efficiently as you can the ring of integers of $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is a fundamental discriminant.

(5) Let $K = \mathbb{Q}(\alpha)$ where $\alpha^3 - \alpha + 1 = 0$. Show that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ and $\mathrm{disc}(K) = -23$. Then compute the Galois closure L of K, and show that only 23 and ∞ ramify in $L/\mathbb{Q}$. Finally, show that L is unramified over $F = \mathbb{Q}(\sqrt{-23})$; what does that tell you about the class group of F and primes of the form $x^2 + xy + 6y^2$?

(6) Let K be a Galois number field and p a prime unramified in K. How do you define the conjugacy class Frob_p? If $K = \mathbb{Q}(\alpha)$ where $f(\alpha) = 0$ and $p \mid \mathrm{disc}(f)$, how does Frob_p relate to the factorization of f modulo p? What does it mean when Frob_p is trivial? How do you define the decomposition group, and what is its relationship to Frob_p?

(7) Define decomposition group and inertia group and relate them by an exact sequence. What are the corresponding orders of these groups in terms of the fundamental invariants?

(8) Let p, q be distinct primes. (You may take $p = 2$ to begin, if you need to.) Let $K = \mathbb{Q}(\zeta_p, \sqrt[p]{q})$. What are the primes (of $\mathbb{Q}$) that ramify in K? What are the inertia groups for these primes?

(9) Define the class group of a Dedekind domain. Compute the class group of $K = \mathbb{Q}(\sqrt{-6})$. Once you know it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, compute the Hilbert class field by showing that $L = K(\sqrt{-3})$ is unramified over K (including at ∞). What are the possible ramification and splitting behaviors of primes in the extension $L/\mathbb{Q}$?

(10) Let p be an odd prime. How do primes factor in $\mathbb{Q}(\zeta_p)$ (according to their congruence class modulo p)? Prove quadratic reciprocity in the form

$$\left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right), \quad \text{where } p^* = \left(\frac{-1}{p}\right) p$$

for primes $q \neq p$ using this factorization and the fact that $\mathbb{Q}(\sqrt{p^*}) \subseteq \mathbb{Q}(\zeta_p)$.

(11) Prove the supplements to quadratic reciprocity using (subfields of) cyclotomic fields.

(12) Show that if K, L are number fields with coprime discriminants $\gcd(d_K, d_L) = 1$, then $K \cap L = \mathbb{Q}$. (How do you know that if $F \supseteq \mathbb{Q}$ has $d_F = 1$ then $F = \mathbb{Q}$?)

(13) Show that if $K \supseteq F \supseteq \mathbb{Q}$ is a tower of number fields, then

$$d_K = d_F^{[K:F]} \operatorname{Nm}_{F/\mathbb{Q}}(\mathfrak{d}_{K/F}).$$

Show that if K has d_K squarefree then K has no nontrivial subfields.

2.2. **Algebraic number theory: local theory.**

(1) Show that $\mathbb{Z}_p$ is a DVR.

(2) Define $\mathbb{Z}_p$ as a projective limit. What is the topology on $\mathbb{Z}_p$ (Hausdorff, compact, and totally disconnected)? Prove that $\mathbb{Z}_p$ is compact and totally disconnected. Give a basis of neighborhoods of 0, and show that open balls are closed.

(3) Find all pure cubic extensions of $\mathbb{Q}_7$. What does this tell you about the abelian extensions?

(4) Exhibit the archimedean places and the places above 2 in the following fields:

$$\mathbb{Q}, \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$$

(Draw a diagram.)

(5) Prove the fundamental identity $n = ef$ for a separable extension L/K of local fields, where e is the ramification degree and f is the inertial degree. Use this to prove the fundamental identity $n = \sum_{i=1}^{r} e_i f_i$ for an extension of global fields.

(6) Prove that the compositum of two unramified extensions of a local field is unramified.

(7) Show that there is a unique unramified extension of a local field in each degree, and show the relative trace and norm are surjective in these extensions.

(8) Give an explicit polynomial $f(x) \in \mathbb{Q}_3[x]$ of degree 4 such that $K = \mathbb{Q}_3[x]/(f(x))$ is a totally ramified quartic extension of $\mathbb{Q}_3$.

2.3. **Analytic number theory.**

(1) Factor the Dedekind zeta function of a quadratic field as the product of two degree 1 L-functions.

(2) What is a Dirichlet character mod N? What is N called? When is it primitive?

(3) What is the prime number theorem, and how does it relate to the zeros of the Riemann zeta function $\zeta(s)$? (What are the trivial zeros of $\zeta(s)$?)

(4) What is the functional equation for ζ and how do you prove it?

3. Algebraic geometry

3.1. **Varieties.**

(1) Consider the image Y of the parametrization

$$\begin{aligned}\mathbb{A}^1 &\to \mathbb{A}^3 \\ t &\mapsto (t, t^2, t^3).\end{aligned}$$

Show that Y is an affine variety and find its defining ideal. (How do you compute the image in general?) What is the projective closure $\overline{Y}$ (the *twisted cubic curve*)? It may help to consider the homogeneous parametrization

$$\begin{aligned}\mathbb{P}^1 &\to \mathbb{P}^3 \\ [s:t] &\mapsto [s^3 : s^2t : st^2 : t^3]\end{aligned}$$

and compute its defining ideal $I(\overline{Y})$. With respect to a few nice term orders, compute the leading terms and initial ideal of $I(\overline{Y})$. What is the Hilbert function and Hilbert polynomial of $I(\overline{Y})$? What do its coefficients tell you?

(2) Consider the quadric surface

$$Q : xw - yz = 0$$

in $\mathbb{P}^3$. Show that it contains the twisted cubic curve. Does Q contain any lines? How are they parametrized? Show that we have an isomorphism of varieties $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Is it true that $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{P}^2$?

(3) Show that the curve $X : y^2 = x^3 + x^2$ over a field k has a singular point at $P = (0,0)$. What are the nicest description you can give of the local ring and the completion of the local ring at P? Compute the normalization of X and describe the birational map between X and its normalization.

3.2. **Schemes.**

(1) Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on a topological space X. What does this mean? Let $P \in X$. Show that $\mathscr{K} = \ker \phi$ is a subsheaf of $\mathscr{F}$, and that $(\ker \phi)_P = \ker(\phi_P)$.

(2) Let X be a scheme and $Z \subseteq X$ and irreducible closed subset. Show that Z corresponds to a generic point of X. (Without loss of generality, you can reduce to the affine case.) Suppose that X is an integral scheme. Show that its function field is the local ring at its generic point.

(3) Is an integral scheme irreducible?

(4) What is a morphism of finite type? What does this mean for X to be a scheme of finite type over $\operatorname{Spec} k$ where k is a field—what do the affine opens look like? What are their closed points if k is algebraically closed? Give a couple of examples of schemes over $\operatorname{Spec} k$ which are not of finite type.