# MATH 250B: COMMUTATIVE ALGEBRA
# HOMEWORK #2

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Some solutions are omitted or sketched.

**Problem XIII.25**. *Let $a_{11}, \dots, a_{1n}$ be elements from a principal ideal ring, and assume that they generate the unit ideal. Show that there exists a matrix $(a_{ij})$ with this given first row, and whose determinant is equal to 1.*

*Solution.* This problem is false for $n = 1$!

Let $R$ be our principal ideal ring. Recall that we denote by $\gcd(x_1, \dots, x_n)$ any element $x$ such that $(x) = (x_1, \dots, x_n)$; such an element is well-defined up to a unit $R^*$.

We prove the result by induction, namely, we show: given $a_{11}, \dots, a_{1n}$, there exists a matrix $A = (a_{ij})$ such that:

(i) $A$ has the given first row;
(ii) $\det A = \gcd(a_{1j})$; and
(iii) If $b_j$ is the minor obtained from the matrix by removing the first row and $j$th column, then $\gcd(b_1, \dots, b_n) = 1$.

First, the base case, $n = 2$. Given $b_1, b_2$ such that $a_{11}b_1 + a_{12}b_2 = \gcd(a_{11}, a_{12})$, we take the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ -b_2 & b_1 \end{pmatrix}.$$

Note that $\gcd(b_1, b_2) = 1$ by the same equation, for $\gcd(b_1, b_2)\gcd(a_{11}, a_{12})$ divides both sides.

Now for the general case: Let $\gcd(a_{11}, \dots, a_{1(n-1)}) = g$, and write $gr + a_{1n}s = \gcd(g, a_{1n})$, (as before $\gcd(r, s) = 1$). By inductive assumption, there exists an $(n-1) \times (n-1)$ matrix $A = (a_{ij})$ with first row $a_{11}, \dots, a_{1(n-1)}$ with $\det A = g$ and $\gcd(b_j) = 1$. Let $b_1c_1 + \dots + b_{n-1}c_{n-1} = 1$; consider the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1(n-1)} & a_{1n} \\ sc_1 & \dots & sc_{n-1} & -r \\ a_{21} & \dots & a_{2(n-1)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{(n-1)1} & \dots & a_{(n-1)(n-1)} & 0 \end{pmatrix}.$$

Note $A$ is obtained from this matrix by removing the second row and last column. By expanding about the last column, we see that the determinant of this matrix is

$$a_{1n}(sc_1b_1 + \dots + sc_{n-1}b_{n-1}) + r\det A = sa_{1n} + rg = \gcd(a_{11}, \dots, a_{1n}).$$

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XIII: 25–28, XIV: 3, 6, 8, 9, 13, 14, 15, 18, 20, 23.

This matrix has the given first row and the correct determinant. To show (iii), we note that the minor obtained by removing the first row and last column is equal to $s$, whereas the minor obtained by removing the first row and $j$th column is equal to $-rb_j$, and $\gcd(-rb_j, s) = \gcd(r, s) = 1$, since $\gcd(b_j) = 1$ by inductive hypothesis. This completes the proof.

**Problem XIII.26**. *Let $A$ be a commutative ring, and $J = (x_1, \dots, x_r)$ an ideal. Let $c_{ij} \in A$ and let*

$$y_i = \sum_{j=1}^{r} c_{ij} x_j.$$

*Let $J' = (y_1, \dots, y_r)$, and let $D = \det(c_{ij})$. Show that $DJ \subset J'$.*

*Solution.* Let $\widetilde{C}$ be the adjoint matrix so that $C\widetilde{C} = DI$. Note that

$$\widetilde{C} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \widetilde{C}C \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = D \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Thus each term $Dx_i = \sum \widetilde{c_{ij}} y_j$, hence $DJ \subset J'$.

**Problem XIII.27**. *Let $L$ be a free module over $\mathbb{Z}$ with basis $e_1, \dots, e_n$. Let $M$ be a free submodule of the same rank, with basis $u_1, \dots, u_n$. Let $u_i = \sum c_{ij} e_j$. Show that the index $(L : M)$ is given by the determinant:*

$$(L : M) = |\det(c_{ij})|.$$

*Solution.* By Theorem III.7.8, there exists a basis $f_1, \dots, f_n$ of $L$ and integers $a_1, \dots, a_n$ such that $a_1 f_1, \dots, a_n f_n$ is a basis of $M$. Write $f_i = \sum p_{ij} e_j$ and $a_i f_i = \sum q_{ij} u_j$; then the matrix $P = (p_{ij})$ is a change of basis for $L$, hence invertible hence $\det P \in \mathbb{Z}^* = \{\pm 1\}$; similarly, $Q = (q_{ij})$ has $|\det Q| = 1$. Then

$$(L : M) = |a_1 \dots a_n| = |\det(QCP^{-1})| = |\det C|.$$

**Problem XIV.3**. *Let $k$ be a commutative ring, and let $M, M'$ be square $n \times n$ matrices in $k$. Show that the characteristic polynomials of $MM'$ and $M'M$ are equal.*

*Solution.* We prove this fact for "generic" matrices $A, A'$, from which the result follows. Let $F$ be the prime subfield of $k$, and consider the polynomial ring $R = F[x_{ij}, x'_{ij}]$ for $1 \le 1, j \le n$. Let $A = (x_{ij})$, $A' = (x'_{ij})$ be matrices of indeterminates, and let $K$ be the field of fractions of $R$, namely, $K = F(x_{ij}, x'_{ij})$. The matrices $M, M'$ are invertible over $K$, since $\det A, \det A' \neq 0$ (they are homogeneous polynomials of degree $n$). Therefore over $K$, we have

$$\det(tI - AA') = \det\left(A^{-1}(tI - AA')A\right) = \det(tI - A'A)$$

so the characteristic polynomials $p_{AA'}(t) = p_{A'A}(t)$. But already $p_{AA'}(t), p_{A'A}(t) \in R[t]$ (clear from the definition as a determinant); we have a map

$$\begin{aligned} R[t] &\to k[t] \\ x_{ij} &\mapsto m_{ij} \\ x'_{ij} &\mapsto m'_{ij} \end{aligned}$$

which sends $p_{AA'}(t) \mapsto p_{MM'}(t)$ and $p_{A'A}(t) \mapsto p_{M'M}(t)$, hence $p_{MM'}(t) = p_{M'M}(t)$.

**Problem XIV.6**. *Let $A$ be a nilpotent endomorphism of a finite-dimensional vector space $E$ over the field $k$. Show that* $\operatorname{tr}(A) = 0$.

*Solution.* The minimal polynomial of $A$ is of the form $t^r$ since $X^m = 0$ for some $m$. Therefore the characteristic polynomial $p_A(t) = t^n$ where $n = \dim E$. Then $\operatorname{tr} A$ is the coefficient of $t^{n-1}$ in $p_A(t)$, namely, $\operatorname{tr} A = 0$.

**Problem XIV.8**. *Let $E$ be a finite-dimensional vector space over a field $k$. Let $A \in \operatorname{Aut}_k(E)$. Show that the following conditions are equivalent:*

(a) *$A = I + N$, with $N$ nilpotent;*
(b) *There exists a basis of $E$ such that the matrix of $A$ with respect to this basis has all its diagonal elements equal to* 1 *and all elements above the diagonal equal to zero.*
(c) *All roots of the characteristic polynomial of $A$ (in the algebraic closure of $k$) are equal to* 1.

*Solution.* If $A = I + N$, then by Exercise 9, there exists a basis of $E$ such that $N$ is strictly upper triangular; let $M$ give such a change of basis, i.e. let $M^{-1}NM$ be strictly upper triangular. If we let $P = P^{-1}$ be the matrix with all anti-diagonal elements equal to 1 and all other elements zero, then $P^{-1}(M^{-1}NM)P$ is strictly lower triangular ($P$ interchanges $i$th and $(n-i)$th rows and columns). Then

$$P^{-1}M^{-1}A(MP) = I + (MP)^{-1}N(MP)$$

so $A$ in this basis has diagonal elements (given by the identity matrix) equal to one, and all elements above the diagonal equal to zero. This shows (a) $\implies$ (b).

The statement (b) $\implies$ (c) is clear, since if we write $A$ in this basis, $A - tI$ is lower triangular so $\det(A - tI) = (t-1)^n$, the product of its diagonal elements.

Finally, to see that (c) $\implies$ (a), we note that $p_A(t) = (t-1)^n$, so $p_A(A) = (A-I)^n = 0$. If we let $N = A - I$, so that $A = I - N$, then $N^n = 0$, i.e. $N$ is nilpotent.

**Problem XIV.9**. *Let $k$ be a field of characteristic* 0 *and let $M$ be an $n \times n$ matrix in $k$. Show that $M$ is nilpotent if and only if* $\operatorname{tr}(M^\nu) = 0$ *for* $1 \le \nu \le n$.

*Solution.* If $M$ is nilpotent, then $M^\nu$ is also nilpotent, hence by Exercise 6 $\operatorname{tr}(M^\nu) = 0$.

Conversely, suppose that $\operatorname{tr}(M^\nu) = 0$. If $\lambda_1, \dots, \lambda_n \in \overline{k}$ are the (not necessarily distinct) eigenvalues of $M$, then $\lambda_1^\nu, \dots, \lambda_n^\nu$ are the eigenvalues of $M^\nu$ (with the same multiplicities), as if $v_i$ has $Mv_i = \lambda v_i$, then

$$M^\nu v_i = M^{\nu-1}(Mv_i) = M^{\nu-1}(\lambda_i v_i) = \cdots = \lambda_i^\nu v_i.$$

It suffices to show that $\lambda_1 = \cdots = \lambda_n = 0$, for then the characteristic polynomial of $M$ is $p_M(t) = \prod (t-\lambda_i)^n = t^n = 0$, i.e. $M$ is nilpotent.

Therefore $\operatorname{tr}(M^\nu) = \sum \lambda_i^\nu$. It suffices to show that $\sum \lambda_i^\nu = 0$ for $1 \le \nu \le n$ implies that $\lambda_1 = \cdots = \lambda_n = 0$. Without loss of generality, we may assume that all $\lambda_i \neq 0$. Consider the Vandermonde matrix

$$V(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & \dots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^n & \dots & \lambda_n^n \end{pmatrix}$$

Since

$$\Lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum \lambda_i \\ \vdots \\ \sum \lambda_i^n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

we see that $\det \Lambda = 0$. But this is a Vandermonde matrix (see after Proposition XIII.4.10), hence

$$\det \Lambda = \lambda_1 \dots \lambda_n \prod_{i<j} (x_j - x_i).$$

Therefore, say, $\lambda_1 = \lambda_2$, so the column vector $(1, -1, 0, \dots, 0)^t$ is in the nullspace of $\Lambda$ and is linearly independent of $(1, \dots, 1)^t$, so the nullspace of $\Lambda$ is at least of dimension 2. This implies that all $(n-1) \times (n-1)$ minors of $\Lambda$ must vanish, in particular, the minor obtained by deleting the first column and row,

$$\lambda_2 \dots \lambda_n \det V(\lambda_2, \dots, \lambda_n) = 0.$$

By induction, this implies that $\lambda_2 = \cdots = \lambda_n = \lambda = \lambda_1$. Since $k$ is characteristic zero, the equation $n\lambda = 0$ implies $\lambda = 0$ as desired.

**Problem XIV.13**. *Let $E$ be a finite-dimensional vector space over a field $k$, and let $S \in \operatorname{End}_k(E)$. We say that $S$ is* diagonalizable *if there exists a basis of $E$ consisting of eigenvectors of $S$. The matrix of $S$ with respect to this basis is then a diagonal matrix.*

(a) *If $S$ is diagonalizable, then its minimal polynomial over $k$ is of type*

$$q(t) = \prod_{i=1}^{m} (t - \lambda_i),$$

*where $\lambda_1, \dots, \lambda_m$ are distinct elements of $k$.*

(b) *Conversely, if the minimal polynomial of $S$ is of the preceding type, then $S$ is diagonalizable. [Hint: The space can be decomposed as a direct sum of the subspaces $E_{\lambda_i}$ annihilated by $S - \lambda_i$.]*

(c) *If $S$ is diagonalizable, and if $F$ is a subspace of $E$ such that $SF \subset F$, show that $S$ is diagonalizble as an endomorphism of $F$, i.e. that $F$ has a basis consisting of eigenvectors of $S$.*

(d) *Let $S, T$ be endomorphisms of $E$, and assume that $S, T$ commute. Assume that both $S, T$ are diagonalizable. Show that they are simultaneously diagonalizable, i.e. there exists a basis of $E$ consisting of eigenvectors for both $S$ and $T$. [Hint: If $\lambda$ is an eigenvalue of $S$, and $E_\lambda$ is the subspace of $E$ consisting of all vectors $v$ such that $Sv = \lambda v$, then $TE_\lambda \subset E_\lambda$.]*

*Solution (sketch).* For (a), let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $S$, and let $v_{ij}$ be a basis of eigenvectors of $S$ with eigenvalues $\lambda_i$. Since already $S - \lambda_i I$ annihilates $v_{ij}$, we see that $\prod_i (S - \lambda_i I)$ annihilates every $v_{ij}$, hence all of $E$, so $\prod_i (S - \lambda_i I) = 0$, and therefore the minimal polynomial of $S$ divides $\prod_i (t - \lambda_i)$, hence is equal to it.

For (b), as in the hint, choose a basis $v_{ij}$ for the subspace $E_{\lambda_i}$; the matrix of $S$ in this basis will be diagonal, since it is diagonal (constant!) on each subspace.

For (c), choose a basis for $F$ and extend it to one of $E$. The matrix for $S$, since $SF \subset F$ is block upper triangular; computing the minimal polynomial for $S$ restricted to $F$ from this we see that it must be squarefree as it divides the minimal polynomial of $S$.

For (d), let $\lambda$ be an eigenvalue of $S$ (such and eigenvalue exists by assumption that $S$ is diagonalizable). In the subspace $E_\lambda = \{v \in S : Sv = \lambda v\}$, we see that

$$S(Tv) = T(Sv) = T(\lambda v) = \lambda(Tv)$$

so $TE_\lambda \subset E_\lambda$. The result now follows from part (c).

**Problem XIV.15**. *Let $E, F$ be finite-dimensional vector spaces over an algebraically closed field $k$, and let $A : E \to E$ and $B : F \to F$ be $k$-endomorphisms of $E, F$, respectively. Let*

$$P_A(t) = \prod (t - \alpha_i)^{n_i}, \qquad P_B(t) = \prod (t - \beta_j)^{m_j}$$

*be the factorizations of their respective characteristic polynomials into distinct linear factors. Then*

$$P_{A \otimes B}(t) = \prod_{i,j} (t - \alpha_i \beta_j)^{n_i m_j}.$$

*[Hint: Decompose $E$ into the direct sum of subspaces $E_i$, where $E_i$ is the subspace of $E$ annihilated by some power of $A - \alpha_i$. Do the same for $F$, getting a decomposition into a direct sum of subspaces $F_j$. Then show that some power of $A \otimes B - \alpha_i \beta_j$ annihilates $E_i \otimes F_j$. Use the fact that $E \otimes F$ is the direct sum of the subspaces $E_i \otimes F_j$, and that $\dim_k(E_i \otimes F_j) = n_i m_j$.]*

*Solution (sketch).* By the hint, it is enough to show that the characteristic polynomial of $A \otimes B$ on $E_i \otimes F_j$ is $(t - \alpha_i \beta_j)^{n_i m_j}$. By direct computation, the only eigenvalue of $A \otimes B$ on $E_i \otimes F_j$ is $\alpha_i \beta_j$. Since we are over an algebraically closed field, this implies the claim.

**Problem XIV.20**.

(a) *How many non-conjugate elements of $GL_2(\mathbb{C})$ are there with characteristic polynomial $t^3(t+1)^2(t-1)$?*
(b) *How many with characteristic polynomial $t^3 - 1001t$?*

*Solution.* Zero and zero, because the degree of the characteristic polynomial is equal to 2. What is up with this problem?

**Problem XIV.23**. *Let $E$ be a finite-dimensional vector space over an algebraically closed field $k$. Let $A, B$ be $k$-endomorphisms of $E$ which commute, i.e. $AB = BA$. Show that $A$ and $B$ have a common eigenvector. [Hint: Consider a subspace consisting of all vectors having a fixed element of $k$ as eigenvalue.]*

*Solution.* Let $\lambda$ be an eigenvector of $A$ ($\lambda$ exists because $k$ is algebraically closed), and let $E_\lambda = \{v \in E : Av = \lambda v\}$. Then for $v \in E_\lambda$,

$$A(Bv) = B(Av) = \lambda Bv$$

so $BE_\lambda \subset E_\lambda$. This implies that $B$ is well-defined as an endomorphism restricted to $E_\lambda$, and therefore (over the algebraically closed field $k$) $B$ has an eigenvector $w \in V$, which by construction is also an eigenvector of $A$.