Problem XVII.12. Prove that an $R$-module $E$ is a generator if and only if it is balanced and finitely generated projective over $\text{End}_R E$.

Solution. Lang proves the $(\Rightarrow)$ direction as Theorem 7.1, so it suffices to show that if $E$ is balanced and finitely generated projective over $\text{End}_R E$, then $R$ is a homomorphic image of a direct sum of $E$ with itself.

Since $E$ is finitely generated projective over $\text{End}_R E$, we have an isomorphism $(\text{End}_R E)^n \cong E \oplus F$ for some $\text{End}_R E$-module $F$. Therefore we have the following isomorphisms of $\text{End}_R E$-modules:

$$E^n \cong \text{Hom}_{\text{End}_R E}((\text{End}_R E)^n, E) \cong \text{Hom}_{\text{End}_R E}(E \oplus F, E) \cong \text{Hom}_{\text{End}_R E}(F, E) \oplus \text{End}_{\text{End}_R E}(E).$$

If we define the operation of $\text{End}_R E$ to be composition of mappings on the left, these become isomorphisms over $R$. Since $E$ is balanced, $\text{End}_{\text{End}_R E}(E) \cong R$, so $E$ is a generator.

Problem X.9(a). Let $A$ be an artinian commutative ring. Prove all prime ideals are maximal. [Hint: Given a prime ideal $p$, let $x \in A$, $x \notin p$. Consider the descending chain $(x) \supset (x^2) \supset \ldots$.]

Solution. We show that any artinian domain is a field. Let $p$ be a prime ideal, so that $A/p$ is a domain. Any quotient ring of an artinian ring is artinian (Proposition 7.1), so $A/p$ is artinian. Let $x \in A/p$ be nonzero. Then the descending chain $(x) \supset (x^2) \supset \ldots$ must terminate, so $(x^k) = (x^{k+1})$ for some integer $k$; therefore there exists a $y \in A/p$ such that $x^{k+1}y = x^k$, which is to say $x^k(1 - xy) = 0$, so $xy = 1$, and $x \in (A/p)^\times$. Therefore $A/p$ is a field, so $p$ is maximal.

Problem X.9(b). There is only a finite number of prime, or maximal, ideals. [Hint: Among all finite intersections of maximal ideals, pick a minimal one.]

Solution. Let $S$ be the set of finite intersections of maximal ideals in $A$. This set is nonempty, so by Exercise XVII.2(c), there exists a minimal such intersection $m_1 \cap \cdots \cap m_r$. If $m$ is any maximal ideal of $A$, then $m \cap \bigcap_i m_i = \bigcap_i m_i$ so $m \supset \bigcap_i m_i \supset m_1 \cap m_2 \cap \cdots m_r$. A maximal ideal is prime, so $m \supset m_i$ for some $i$, but since $m_i$ is maximal so $m = m_i$. 

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XVII: 12 (first sentence); X: 9, 10, 11.
Problem X.9(c). The ideal $N$ of nilpotent elements in $A$ is nilpotent, that is there exists a positive integer $k$ such that $N^k = (0)$. [Hint: Let $k$ be such that $N^k = N^{k+1}$. Let $a = N^k$. Let $b$ be a minimal ideal such that $ba \neq 0$. Then $b$ is principal and $ba = b$.]

Solution. Let $k$ be such that $N^k = N^{k+1}$. Suppose that $N^k \neq 0$; let $S$ be the set of ideals $b$ of $A$ such that $bN^k \neq 0$. The set $S$ is nonempty because $N^k \in S$, as $N^kN^k = N^{2k} = N^k \neq 0$. Since $A$ is artinian, $S$ has a minimal element $b$. There is an element $b \in b$ such that $bN^k \neq 0$; therefore $(b) = b$, in particular, $b$ is finitely generated. But $bN^k \subset b$ and $(bN^k)N^k = bN^{2k} = bN^k$, so again by minimality, $bN^k = b$. But every element of $N^k$ is nilpotent hence contained in every maximal ideal, so by Nakayama’s lemma, $b = 0$, a contradiction.

Problem X.9(d). $A$ is noetherian.

Solution. Let $k$ be an integer such that $N^k = 0$ as in part (c). Then $\bigcap_i m_i = N$ = $N$, but by part (a) this implies $N = \bigcap_i m_i$. Let $k$ be an integer such that $N^k = 0$. Then

$$N^k = 0 = (\bigcap_i m_i)^k = (m_1 \ldots m_r)^k = m_1^k \ldots m_r^k.$$

Consider $A$ as a module over itself; $A$ is noetherian as an $A$-module if and only if $A$ is noetherian as a ring. We have a filtration

$$A \supset m_1 \supset m_2^2 \supset \cdots \supset m_i^k \ldots m_r^k = 0$$

of $A$. At the step $E \supset E m_i$ in the filtration, $E/E m_i$ is a vector space over the field $A/m_i$, which is finite-dimensional as $A$ is artinian (as in Exercise XVII.2(a)). Therefore $A$ has a finite simple filtration, so by Proposition 7.2, $A$ is noetherian as well as artinian.

Problem X.9(e). There exists an integer $r$ such that

$$A \cong \prod_m A/m_i^r$$

where the product is taken over all maximal ideals.

Solution. Let $r$ be such that $N^r = 0$, and let $m_i$ be the maximal ideals of $A$. Since $m_i + m_j = A$ for $i \neq j$, we also have $m_i^r + m_j^r = A$ for $i \neq j$ (otherwise, $m_i^r + m_j^r \subset m$ for some maximal ideal $m$; then $m_i^r \subset m$ so $m_i \subset m$, and similarly $m_j \subset m$, a contradiction). By the Chinese remainder theorem, then,

$$A \rightarrow \prod_m A/m_i^r$$

is surjective. It is also injective, since $\bigcap_i m_i^r = N^r = 0$ (as in part (d)), therefore it is an isomorphism.

Problem X.9(f). We have

$$A \cong \prod_p A_p$$

where again the product is taken over all prime ideals $p$. 


Solution. It is enough to show that this map is an isomorphism considered as a map of \( A \)-modules. Let \( p_i \) be the primes (maximal ideals) of \( A \). Since localization preserves exact sequences (it is flat), it is enough to show that the map \( A \rightarrow \prod_i A_{p_i} \) is an isomorphism after localization at every prime ideal \( p \) of \( A \). But in this circumstance we have the map

\[
A_p \rightarrow \prod_i (A_{p_i})_p = \prod_i (A_{p_i})_{p_i}.
\]

Now \( A_p \) is artinian (descending chains of ideals of \( A_p \) are descending chains of ideals of \( A \) contained in \( p \)) and a local ring with maximal ideal \( p A_p \), so \( N = p A_p \) and \( (p A_p)^r = 0 \). Then for \( p \neq p_i \), if \( x_i \in p \setminus p_i \), \( x_i^r = 0 \), so \( (A_{p_i})_p = 0 \) and the map is an isomorphism.

Problem X.10. Let \( A, B \) be local rings with maximal ideals \( m_A, m_B \), respectively. Let \( f : A \rightarrow B \) be a homomorphism. Suppose that \( f \) is local, i.e. \( f^{-1}(m_B) = m_A \).

Assume that \( A, B \) are noetherian, and assume that:

1. \( A/m_A \rightarrow B/m_B \) is an isomorphism;
2. \( m_A \rightarrow m_B/m_B^2 \) is surjective;
3. \( B \) is a finite \( A \)-module, via \( f \).

Prove that \( f \) is surjective.

Solution. First, \( m_B \) is a finitely generated \( B \)-module (since \( B \) is noetherian) and \( f(m_A) \) is a finitely generated \( B \)-submodule of \( m_B \) with \( m_B = f(m_A) + m_B^2 \) by (2). By Nakayama’s lemma (X.4.2), \( f(m_A) = m_B \).

Second, since \( B \) is finite over \( A \) and \( f(A) \) is a \( B \)-submodule with \( B = f(A) + m_BB \) by (1), since \( A \rightarrow B/m_B \) is surjective. But \( m_BB = m_AB \) treating \( B \) as an \( A \)-module by \( f \), so by Nakayama’s lemma, \( f(A) = B \), so \( f \) is surjective.

Problem X.11. Let \( A \) be a commutative ring and \( M \) an \( A \)-module. Define the support of \( M \) by

\[
\text{Supp}(M) = \{ p \in \text{Spec} \ A : M_p \neq 0 \}.
\]

If \( M \) is finite over \( A \), show that \( \text{Supp}M = V(\text{Ann}(M)) \), where \( V(a) = \{ p \in \text{Spec} \ A : p \supset a \} \) and \( \text{Ann}(M) = \{ a \in A : aM = 0 \} \).

Solution. Let \( p \in \text{Spec} \ A \) be so that \( M_p \neq 0 \). If \( aM = 0 \) then \( aM_p = 0 \) so if \( a \notin p \) then \( M_p = 0 \); hence \( \text{Ann}(M) \subset p \).

Conversely, suppose \( \text{Ann}(M) \subset p \). Let \( m_1, \ldots, m_r \) generate \( M \) over \( A \). Suppose that \( M_p = 0 \); then for all \( i \) there exists an \( a_i \notin p \) such that \( a_im_i = 0 \in M \). Then \( a = \prod_i a_i \) has \( aM = 0 \); therefore \( a \in p \), a contradiction.