MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #3A

Problem 1 (DF 11.1.8, 11.1.9, 11.2.8). Let V be a vector space over F and let $\phi : V \to V$ be a linear transformation. A nonzero element $v \in V$ satisfying $\phi(v) = \lambda v$ for some $\lambda \in F$ is called an *eigenvector* of ϕ with *eigenvalue* λ.

(a) Prove that for any fixed $\lambda \in F$, the collection of eigenvectors of ϕ with eigenvalue λ, together with 0, forms a subspace of V.

(b) Suppose for $i = 1, \dots, k$ that $v_i \in V$ is an eigenvector of ϕ with eigenvalue λ_i and that all of the eigenvalues λ_i are distinct. Prove that $v_1, \dots, v_k$ are linearly independent. Conclude that any linear transformation on an n-dimensional vector space has at most n distinct eigenvalues.

(c) Prove that if V has a basis consisting of eigenvectors of ϕ, then the matrix representing ϕ with respect to this basis is diagonal. What are the diagonal entries?

(d) Prove that an $n \times n$ matrix A is similar to a diagonal matrix if and only if V has a basis of eigenvectors for $L(A)$.

Problem 2. Let $\phi : V \to V$ be a linear transformation over a field F and let β be a basis of V.

(a) Show that ϕ is invertible if and only if ϕ maps β to a basis of V if and only if the column vectors of $M(\phi)_\beta$ are a basis of V.

(b) Suppose that $\#F = q$. Show that

$$\#GL_n(V) = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}).$$

Problem 3 (sorta DF 11.2.35).

(a) Define the *trace* map

$$\begin{aligned} \mathrm{tr} : M_2(F) &\to F \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto a + d. \end{aligned}$$

Show that tr is a linear transformation and determine the matrix of tr with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_2(F)$.

(b) Generalize part (a) to $\mathrm{tr} : M_n(F) \to F$ for arbitrary $n \in \mathbb{Z}_{>0}$.

Date: 11 February 2008; due Friday, 22 February 2008.