MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #3B

Problem 4 (sorta DF 11.1.5). Let $a, b \in \mathbb{R}$ with $a < b$. Let V denote the space of real-valued functions on the closed interval $[a, b]$.

(a) Show that V is an infinite dimensional vector space over $\mathbb{R}$, and hence is isomorphic to an (uncountably) infinite direct product of copies of $\mathbb{R}$.

(b) Let $C([a, b]) \subset V$ denote the subspace of continuous functions. Show that for any $g \in C([a, b])$, the function $\phi_g : V \to \mathbb{R}$ defined by $\phi_g(f) = \int_a^b f(t)g(t)\,dt$ is a linear functional on $C([a, b])$.

(c) Let
$$W = \mathbb{R}[x]_{\leq 2} = \{a_2x^2 + a_1x + a_0 : a_i \in \mathbb{R}\} \subset C([a, b]).$$
Let $\beta = \{1, x, x^2\}$. For each $f^* \in \beta^* \subset W^*$, find a $g(x) \in W$ such that $f^* = \phi_g$.

Problem 5 (DF 11.3.4–5). Let V be a vector space with basis β.

(a) Show that V^* is isomorphic to the direct product of copies of F indexed by β.

(b) If $\#\beta = \infty$, show that β^* does *not* span V^*, hence $\dim V^* > \dim V$.

Problem 6. Let R be a commutative ring.

(a) Prove that for all $A, B \in M_n(R)$, we have $\det(AB) = \det(A)\det(B)$. *[Hint: Feel free to reproduce the one in the book, if you can follow it, or any other from your favorite linear algebra book if it applies over a ring R.]*

(b) Prove that $A \in M_n(R)$ is invertible if and only if $\det(A) \in R^\times$.

Date: 18 February 2008; due Friday, 29 February 2008.