MATH 255: ELEMENTARY NUMBER THEORY
HOMEWORK #1

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§1.1: Numbers, Sequences, and Sums

Problem 1.1.1. Determine whether each of the following sets is well-ordered. If so, what is its least element?

(a) The set of integers greater than 3.
(b) The set of even positive integers.
(c) The set of positive rational numbers.
(d) The set of positive rational numbers that can be written in the form $a/2$, where a is a positive integer.
(e) The set of nonnegative rational numbers.

Problem 1.1.4. For each of the following statements, either give a proof or a counterexample.

(a) The sum of a rational and an irrational number is irrational.
(b) The sum of two irrational numbers is irrational.
(c) The product of a rational number and an irrational number is irrational.
(d) The product of two irrational numbers is irrational.

Problem 1.1.7(a)(c)(e). Find the following values of the floor function.

(a) $\lfloor 1/4 \rfloor$.
(c) $\lfloor 22/7 \rfloor$.
(e) $\lfloor \lfloor 1/2 \rfloor + \lfloor 1/2 \rfloor \rfloor$.

§1.2: Sums and Products

Problem 1.2.8. The *pentagonal numbers* $p_1, p_2, \dots, p_k, \dots$ are the integers that count the number of dots in k nested pentagons. Show that $p_1 = 1$ and $p_k = p_{k-1} + (3k-2)$ for $k \geq 2$. Conclude that $p_n = \sum_{k=1}^{n} (3k-2)$.

Problem 1.2.9. Prove that the sum of the $(n-1)$th triangular number and the nth square number is the nth pentagonal number.

Date: Due Wednesday, 21 January 2009.

§1.3: Mathematical Induction

Problem 1.3.4. Conjecture a formula for

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

from the value of this sum for small integers n. Prove that your conjecture is correct using mathematical induction.

§1.5: Divisibility

Problem 1.5.6. What can you conclude if a and b are nonzero integers such that $a \mid b$ and $b \mid a$?

Problem 1.5.7. Show that if $a, b, c, d \in \mathbb{Z}$ with a, c nonzero, such that $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Problem 1.5.8. Are there $a, b, c \in \mathbb{Z}$ such that $a \mid bc$ but $a \nmid b$ and $a \nmid c$?

Problem 1.5.24. Find the number of positive integers not exceeding 1000 that are not divisible by 3 or 5.

Problem 1.5.34. Use mathematical induction to show that $n^5 - n$ is divisible by 5 for every positive integer n.