MATH 255: ELEMENTARY NUMBER THEORY
HOMEWORK #3

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§3.3: Greatest Common Divisors

Problem 3.3.6. Let a be a positive integer. What is the greatest common divisor of a and $a+2$?

Problem 3.3.7. Show that if a and b are integers, not both zero, and c is a nonzero integer, then $\gcd(ca, cb) = |c| \gcd(a, b)$.

Problem 3.3.8. Show that if a and b are integers with $\gcd(a, b) = 1$, then $\gcd(a+b, a-b) = 1$ or 2.

Problem 3.3.32. What can you conclude if $a, b, c \in \mathbb{Z}_{>0}$ satisfy $\gcd(a, b) = \gcd(b, c) = 1$ and $1/a + 1/b + 1/c$ is an integer? *[Hint: Consider their sizes!]*

§3.4: The Euclidean Algorithm

Problem 3.4.1. Use the Euclidean algorithm to find each of the following greatest common divisors.

(a) $\gcd(45, 75)$.
(c) $\gcd(666, 1414)$.

Problem 3.4.3. For each pair of integers in Exercise 1(a)(c), express the greatest common divisor of the integers as a linear combination of these integers.

§3.5: The Fundamental Theorem of Arithmetic

Problem 3.5.2. Find the prime factorization of 111 111.

Problem 3.5.4(d). Find all prime factors of the integer $\binom{30}{10}$.

Problem 3.5.7. Which integers have exactly three positive divisors? Which have exactly four positive divisors?

Problem 3.5.10. Show that if a and b are positive integers and $a^3 \mid b^2$, then $a \mid b$.

Problem 3.5.44. Show that $\sqrt[3]{5}$ is irrational:

(a) By an argument similar to that given in Example 3.20;
(b) Using Theorem 3.18.

Date: Due Wednesday, 4 February 2009.

Problem 3.5.A. Let E be the set of positive even integers. We will show that E does not have unique factorization.

(a) Show that the set E is closed under multiplication: i.e., if $a, b \in E$ then $ab \in E$.

(b) An integer $n \in E$ is *E-composite* if $n = ab$ for some $a, b \in E$; an integer $n \in E$ is *E-prime* if n is not composite. Of the integers $2, 4, 6, \ldots, 20$, which are E-prime and which are E-composite? Give a necessary and sufficient condition for $n \in E$ to be E-prime.

(c) Show that there exists $n \in E$ which can be factored into E-primes in two different ways.